1. Suppose we have samples $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$, both of which come from a Normal distribution with common variance $\sigma^2$ but possibly different means $\mu_X$ and $\mu_Y$. Suppose you wanted to test $H_0 : \mu_X = 2\mu_Y$ versus the two-sided alternative.

a. Define the random variable $W = \bar{X} - 2\bar{Y}$. How is $W$ distributed (be sure to include the values of any parameters)?

First, $W$ is normally distributed since it is a linear combination of normal random variables. It’s expected value is

$$\mathbb{E} W = \mathbb{E} (\bar{X} - 2\bar{Y}) = \mu_X - 2\mu_Y$$

and variance

$$\text{Var} (W) = \text{Var} (\bar{X} - 2\bar{Y}) = \frac{\sigma^2}{n} + 4 \frac{\sigma^2}{m},$$

hence $W \sim N(\mu_X - 2\mu_Y, \sigma^2 (\frac{1}{n} + \frac{4}{m}))$.

b. Assuming $\sigma$ to be known, find a $(1 - \alpha)\%$ confidence interval for $W$.

Using part a, we have

$$W \pm z_{\alpha/2}\sigma_W$$

or more specifically

$$\bar{X} - 2\bar{Y} \pm z_{\alpha/2}\sigma\sqrt{\frac{1}{n} + \frac{4}{m}}$$

c. Assuming $\sigma$ is NOT known, find a $(1 - \alpha)\%$ confidence interval for $W$.

(Hint: let $s_p$ denote the approximation for $\sigma$)

We have similarly

$$\bar{X} - 2\bar{Y} \pm t_{m+n-2,\alpha/2}s_p\sqrt{\frac{1}{n} + \frac{4}{m}}$$

2. Respond to the following: Problem 1 is pointless. All I need to do is look at the observed values of $\bar{x}$ and $2\bar{y}$ and see if they’re equal.

Ans: The test is whether $\mu_X = 2\mu_Y$, and indeed the samples will certainly never have $\bar{x} - 2\bar{y}$, and so we are instead looking at significant deviations from this assumption, which takes the form of large values of $|\bar{x} - 2\bar{y}|$ since we are likely to have small variations, but large enough variations will give significant evidence to the contrary of the null.