1. Suppose \( X_1, X_2, \ldots, X_n \) are independent normal random variables with mean \( \mu \) and variance \( \sigma^2 \). Let \( W_n = \sum_{i=1}^{n} \left( \frac{X_i - \mu}{\sigma} \right)^2 \). What is the distribution of \( W_n \)?

Solution.

\[
\left( \frac{X_i - \mu}{\sigma} \right) = Z \sim N(0, 1)
\]

so

\[
W_n = \sum_{i=1}^{n} Z_i^2 \sim \chi^2_n
\]

i.e. \( W_n \) is the sum of \( n \) standard normals squared, so it is chi-squared with \( n \) degrees of freedom.

2. Let \( X_1, \ldots, X_n \) be a sample from an \( N(\mu_X, \sigma_X^2) \) distribution and \( Y_1, \ldots, Y_m \) be an independent random sample from a \( N(\mu_Y, \sigma_Y^2) \) distribution. Show how to use the F distribution to find \( P\left( \frac{S_X^2}{S_Y^2} > c \right) \) (You may leave your answer in the form of a probability involving a random variable with F distribution, but be sure to include the degrees of freedom).

(HINT):

a. \((n-1)S_X^2/\sigma_X^2\) has a \( \chi^2_{n-1} \) distribution with \( n - 1 \) degrees of freedom.

b. The random variable \( W \) has F distribution with \( m \) and \( n \) degrees of freedom if

\[
W = \frac{U/m}{V/n},
\]

where \( U \) and \( V \) are independent chi-square random variables with \( m \) and \( n \) degrees of freedom, respectively.)

From part a of the hint, we have

\[
\frac{(n-1)S_X^2}{\sigma_X^2} \sim \chi^2_{n-1}
\]

\[
\frac{(m-1)S_Y^2}{\sigma_Y^2} \sim \chi^2_{m-1}.
\]

From part b of the hint we know that

\[
\frac{(n-1)S_X^2/(n-1)}{(m-1)S_Y^2/(m-1)} = W \sim F_{n-1,m-1}.
\]

Now we simplify \( W \) to obtain

\[
W = \frac{S_X^2/\sigma_X^2}{S_Y^2/\sigma_Y^2}.
\]
hence

\[ P \left( \frac{S_X^2}{X^2_1} > c \right) = P \left( \frac{S_X^2 \sigma_Y^2}{S_Y^2 \sigma_X^2} > c \frac{\sigma_Y^2}{\sigma_X^2} \right) \]

\[ = P \left( W > c \frac{\sigma_Y^2}{\sigma_X^2} \right), \]

where \( W \sim F_{n-1, m-1} \).