

Explicit Inner and Outer Bounds for Multi-source Multi-sink Network Coding

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Abstract—In multi-source multi-sink network coding, messages across different sources are coded to increase the overall throughput. The various types of coded information in the network significantly complicate the determination of its capacity region. In this work, we derive explicit inner and outer bounds for acyclic multi-source multi-sink networks based on a cut-based network decomposition technique and a role-based information characterization technique. In particular, we derive a linear programming inner bound for regular K -pairs acyclic three-layer networks and a network sharing outer bound for arbitrary acyclic multi-source multi-sink networks. The techniques used in this paper reveal some of the basic mechanisms of multi-source multi-sink network coding.

I. INTRODUCTION

Unlike messages that are coded within the same source and requested at all sinks in single-source multicast networks, messages in multi-source multi-sink networks are coded across different sources and requested at possibly different sinks. The various types of information flows from cross coding significantly complicate the decoding at the sinks. For this reason, the max-flow min-cut bound, which characterizes the capacity region for single-source multicast networks [3] can not be simply extended to the multi-source multi-sink networks. In fact, it only gives a loose outer bound in general [4], [2]. Therefore, it is an interesting problem to characterize these information flows and find the underlying inner and outer bounds.

In this paper, we derive a linear programming inner bound for regular K -pairs acyclic three-layer networks and a network sharing outer bound for general K -pairs acyclic multi-source multi-sink networks based on a cut-based network decomposition technique and a role-based information characterization technique proposed in [1] and [2]. We further prove that the network sharing outer bound can be extended to arbitrary acyclic multi-source multi-sink networks via a sink decomposition technique.

II. NETWORK MODEL

A. Some Notations

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ denote an acyclic directed graph, where \mathcal{V} is the set of nodes and \mathcal{E} is the set of directed edges. A pair $(i, j) \in \mathcal{E}$ is an edge directed from node i to node j . Let

$$\mathcal{A}_j = \{i \in \mathcal{V}, (i, j) \in \mathcal{E}\}, \mathcal{B}_j = \{i \in \mathcal{V}, (j, i) \in \mathcal{E}\}, \quad (1)$$

and assume that $\mathcal{A}_{s_j} = \phi$ for any source node s_j and $\mathcal{B}_{t_j} = \phi$ for any sink node t_j . We define a *path* $p(i_0, i_l)$ in \mathcal{G} to be a sequence of distinct ordered nodes (i_0, i_1, \dots, i_l) , where $(i_{j-1}, i_j) \in \mathcal{E}, 1 \leq j \leq l$. For acyclic networks, a path consists of distinct nodes. For an acyclic directed graph, define $i \prec j$ if there exists a path from i to j . The relation \prec is a *partial order*.

B. A Model of Acyclic Multi-source Multi-sink Networks

Definition 1: A multi-source multi-sink network with a set of transmission requirements \mathcal{D} is specified by the following elements:

- 1) \mathcal{S} , the set of source nodes $\{s_1, \dots, s_{|\mathcal{S}|}\}$;
- 2) \mathcal{T} , the set of sink nodes $\{t_1, \dots, t_{|\mathcal{T}|}\}$;
- 3) $\mathcal{C} = (C_{ij} : (i, j) \in \mathcal{E})$, the set of capacity constraints;
- 4) $\mathcal{D} = \{\mathcal{D}_i : t_i \in \mathcal{T}\}$, where $\mathcal{D}_i \in 2^{\mathcal{S}} \setminus \phi$, the reconstruction requirements of the sink nodes.

Definition 2: A K -pairs multi-source multi-sink network is a multi-source multi-sink network with $|\mathcal{S}| = |\mathcal{T}| = K$ and $\mathcal{D}_i = \{s_i\}$.

We assume the sources $\mathbf{X}_j = \{X_{jk}\}_{k=1}^{\infty}$ generated at source nodes $s_j, j = 1, \dots, |\mathcal{S}|$ are independent, where $X_{jk}, k = 1, 2, \dots$ are independently and identically distributed copies of a generic random variable X_j with alphabet $\mathcal{X}_j < \infty$.

We define a *cut* \mathcal{U} in $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ to be a subset $\mathcal{U} \subseteq \mathcal{V}$. For a cut \mathcal{U} we define

$$\mathcal{E}_{\mathcal{U}} = \{(i, j) \in \mathcal{E} : i \in \mathcal{U}, j \in \mathcal{V} \setminus \mathcal{U}\}. \quad (2)$$

Suppose that a cut satisfies the condition that $\mathcal{S}_{\alpha} = \{s_i \in \mathcal{S} : i \in \alpha\} \subseteq \mathcal{U}$ and $\mathcal{T}_{\beta} = \{t_i \in \mathcal{T} : i \in \beta\} \subseteq \mathcal{V} \setminus \mathcal{U}$, then we say $\mathcal{U} \in \mathcal{U}_{\alpha, \beta}$, where $\mathcal{U}_{\alpha, \beta}$ represents the set of all \mathcal{U} that separate the source subset \mathcal{S}_{α} and the sink subset \mathcal{T}_{β} . Define the *cut set* of $\mathcal{U} \in \mathcal{U}_{\alpha, \beta}$ as

$$\mathcal{E}_{\mathcal{U}}(\alpha, \beta) = \{(i, j) \in \mathcal{E}(\alpha, \beta) : i \in \mathcal{U} \text{ and } j \notin \mathcal{U}\}, \quad (3)$$

where $\mathcal{E}(\alpha, \beta) = \{(i, j) \in \mathcal{E} : \exists k \in \alpha \text{ such that } s_k \prec i,$

$$\exists l \in \beta \text{ such that } j \prec t_l\}, \quad (4)$$

and the capacity of $\mathcal{U} \in \mathcal{U}_{\alpha, \beta}$ as

$$C_{\mathcal{U}}(\alpha, \beta) = \sum_{(i, j) \in \mathcal{E}_{\mathcal{U}}(\alpha, \beta)} C_{ij}. \quad (5)$$

C. Admissible Rate Region and Capacity Region for Multi-source Multi-sink Networks

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a multi-source multi-sink network with a set of transmission requirements \mathcal{D} , we define

$$d_i : \left(\prod_{j \in \mathcal{D}_i} \mathcal{X}_j \right) \times \left(\prod_{j \in \mathcal{D}_i} \mathcal{X}_j \right) \rightarrow \{0, 1\}$$

as the Hamming distortion measure for $t_i \in \mathcal{T}$, i.e., for any \mathbf{x} and \mathbf{x}' in $(\prod_{j \in \mathcal{D}_i} \mathcal{X}_j) \times (\prod_{j \in \mathcal{D}_i} \mathcal{X}_j)$,

$$d_i(\mathbf{x}, \mathbf{x}') = \begin{cases} 0, & \text{if } \mathbf{x} = \mathbf{x}' \\ 1, & \text{if } \mathbf{x} \neq \mathbf{x}' \end{cases}$$

Let $X_j^n = (X_{j1}, \dots, X_{jn})$. We consider an $(n, (\eta_{ij}, (i, j) \in \mathcal{E}), (\Delta_l, l \in \mathcal{T}))$ block code of block length n consisting of

- 1) for each $(i, j) \in \mathcal{E}$, an encoding function

$$F_{ij} : \prod_{i' \in \mathcal{S}: (i', i) \in \mathcal{E}} \mathcal{X}_{i'}^n \times \prod_{i' \notin \mathcal{S}: (i', i) \in \mathcal{E}} \{0, 1, \dots, \eta_{i'i} - 1\} \rightarrow \{0, 1, \dots, \eta_{ij} - 1\}; \quad (6)$$

- 2) for each $l \in \mathcal{T}$, a decoding function

$$G_l : \prod_{v \in \mathcal{S}: (v, l) \in \mathcal{E}} \mathcal{X}_v^n \times \prod_{v \notin \mathcal{S}: (v, l) \in \mathcal{E}} \{0, 1, \dots, \eta_{vl} - 1\} \rightarrow \prod_{k \in \mathcal{D}_l} \mathcal{X}_k^n; \quad (7)$$

- 3) for each $l \in \mathcal{T}$, an average distortion

$$\Delta_l = n^{-1} E \sum_{k=1}^n d_l((X_{jk}, j \in \mathcal{D}_l), (\hat{X}_{jk}, j \in \mathcal{D}_l)), \quad (8)$$

where $(\hat{X}_j^n, j \in \mathcal{D}_l)$ are the outputs of the decoder G_l when the inputs are $X_j^n, j = 1, \dots, |\mathcal{S}|$ and E is the expectation over the source input distribution.

A rate vector $\mathbf{R} = (R_{ij}, (i, j) \in \mathcal{E})$ is admissible if for every $\epsilon > 0$, there exists for sufficiently large n an $(n, (\eta_{ij}, (i, j) \in \mathcal{E}), (\Delta_l, l \in \mathcal{T}))$ code such that

$$n^{-1} \log \eta_{ij} \leq R_{ij}, \forall (i, j) \in \mathcal{E} \text{ and } \Delta_l \leq \epsilon, \forall l \in \mathcal{T}.$$

The capacity region \mathcal{H} is defined as the region of all entropy vectors $\mathbf{H} = (H(X_k) : s_k \in \mathcal{S})$ for which the capacity vector \mathbf{C} is admissible.

D. The Cut-based Network Decomposition and The Role-based Information Characterization Techniques

1) *The Cut-based Network Decomposition.* A three-layer network, as defined in [1], is a special network whose coding channels represent a cut in a general multi-source multi-sink network. Therefore, any arbitrary acyclic multi-source multi-sink network can be decomposed into a set of three-layer networks when cuts are taken to provide constraints on the total amount of information in transmission. An example of a three-layer network is shown in Fig. 1, where channels (i, i') are the coding channels representing a cut and channels (S_j, i)

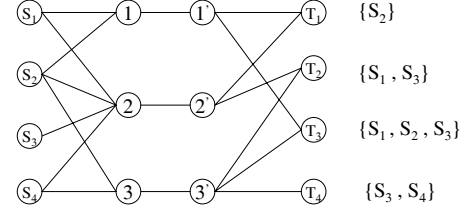


Fig. 1. An example of a three-layer network, where channels (i, i') are the only (coding) channels with capacity constraints.

and (i', T_k) are direct channels representing paths from S_j to i and paths from i' to T_k respectively in a general multi-source multi-sink network.

2) *The Role-based Information Characterization.* As described earlier, the various types of cross coded information in a multi-source multi-sink network increase coding complexity. However, it is not hard to differentiate them by the roles they play in coding. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a given multi-source multi-sink network. For any $(i, j) \in \mathcal{E}$, define $\mathcal{S}_{ij} = \{k : s_k \in \mathcal{S}, s_k \prec i\}$ and $\mathcal{T}_{ij} = \{l : t_l \in \mathcal{T}, j \prec t_l\}$, we have the following definitions:

Definition 3: For any $(i, j) \in \mathcal{E}$, (i, j) is said to be a *main channel* and its output is called *main information* if $\mathcal{S}_{ij} = \mathcal{T}_{ij}$; (i, j) is said to be a *side information channel* and its output is called *side information* if $\mathcal{S}_{ij} \cap \mathcal{T}_{ij} = \phi$; (i, j) is said to be an *irregular channel* and its output is called *irregular information* if $\mathcal{S}_{ij} \cap \mathcal{T}_{ij} \neq \phi$ and $\mathcal{S}_{ij} \neq \mathcal{T}_{ij}$.

Definition 4: A *regular three-layer network* is a three-layer network whose coding channels consist of only main channels and side information channels. It is essentially a three-layer network with arbitrary degree of cross coding [2].

III. MAIN RESULTS

A. A Linear Programming Inner Bound for Regular Acyclic K -pairs Three-layer Networks via Matrix Tiling

In a regular K -pairs acyclic three-layer networks, there are only two types of coding channels, i.e., the main channels and the side information channels. Index the channels (i, j) by $(\mathcal{S}_{ij}, \mathcal{T}_{ij})$ and the capacity C_{ij} by $C_{\mathcal{S}_{ij}}^{\mathcal{T}_{ij}}$. If two channels have the same indexing, then they are viewed as a single channel with capacity being the sum of the two capacities. Therefore, by enumerating all possible ways of assigning side information to the main channels, we obtain all possible source entropy rates of linear coding. In fact, this rate assigning problem is equivalent to the maximum weight matrix tiling problem and can be readily solved through a linear programming approach. This is seen through the following observations:

- 1) Each main channel (λ, λ) can be regarded as a supplier that supplies diagonal tiles with rows and columns indexed by $\delta \subseteq \lambda$ with weights $R_\lambda^\lambda(\delta)$. The sum of these weights is upper bounded by the capacity of (λ, λ) .
- 2) Each side information channel $(\alpha, \beta), \alpha \neq \beta$ can be regarded as a supplier that supplies rectangular tiles with rows and columns indexed by α and β whose weights are

$R_\alpha^\beta(\delta)$ when used for matrices $\delta, \alpha \cap \delta \neq \phi, \beta \cap \delta \neq \phi$. The sum of these weights is upper bounded by the capacity of (α, β) .

Therefore, the problem is to find an optimal tiling solution that obtains the maximum sum weight of the tiled matrices subject to the constraints of the supplies of the tiles.

Theorem 1: For any regular K -pairs acyclic three-layer networks, let $\mathcal{I} = \{1, 2, \dots, K\}$ be the index set of both \mathcal{S} and \mathcal{T} , for any nonempty set $\gamma \subseteq \mathcal{I}$, we have

$$\sum_{i \in \gamma} H(\mathcal{X}_i) \geq M(\gamma) \quad (9)$$

where

$$M(\gamma) = \max_{\lambda \subseteq \mathcal{I}: \lambda \cap \gamma \neq \phi} \sum_{\delta \subseteq \lambda \cap \gamma} |\delta| R_\lambda^\lambda(\delta) \quad (10)$$

Subject to:

$$\sum_{\delta \subseteq \lambda \cap \gamma} R_\lambda^\lambda(\delta) \leq C_\lambda^\lambda, \forall \lambda \subseteq \mathcal{I}, \lambda \cap \gamma \neq \phi, \quad (11)$$

$$\sum_{\delta \subseteq \gamma: \substack{\alpha \cap \delta \neq \phi, \\ \beta \cap \delta \neq \phi}} R_\alpha^\beta(\delta) \leq C_\alpha^\beta, \forall \alpha \subseteq \mathcal{I}, \beta \subseteq \mathcal{I}, \alpha \cap \beta = \phi, \quad (12)$$

$$\sum_{(\alpha, \beta): \substack{(x, y) \in \\ (\alpha \times \beta) \cap (\delta \times \delta)}} R_\alpha^\beta(\delta) \geq \sum_{\substack{\lambda \subseteq \mathcal{I}: \\ \lambda \supseteq \delta}} R_\lambda^\lambda(\delta), \forall \delta \subseteq \gamma, (x, y) \in (\delta \times \delta). \quad (13)$$

B. A Network Sharing Outer Bound for Arbitrary K -pairs Acyclic Multi-source Multi-sink Networks

Suppose that $\gamma \subseteq \{1, 2, \dots, K\}, \gamma \neq \phi$ and \prec_γ is a linear order in γ . Assuming $\gamma = \{i_1 \prec_\gamma \dots \prec_\gamma i_{|\gamma|}\}$, we define

$$i_\beta \triangleq \min_{\prec_\gamma} \{\beta \cap \gamma\}, \quad (14)$$

$$\gamma(\beta) \triangleq \{i \in \gamma : i \prec_\gamma i_\beta, i \neq i_\beta\}, \quad (15)$$

$$\mathcal{U}_\gamma \triangleq \{\mathcal{U} : \mathcal{S}_\gamma \subseteq \mathcal{U}, \mathcal{T}_\gamma \subseteq \mathcal{V} \setminus \mathcal{U}\}, \quad (16)$$

where $\mathcal{S}_\gamma = \{s_i : i \in \gamma\}, \mathcal{T}_\gamma = \{t_i : i \in \gamma\}, \beta \subseteq \mathcal{I}, \beta \neq \phi$.

Theorem 2: Given a K -pairs acyclic multi-source multi-sink network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with $\mathcal{I} = \{1, \dots, K\}$, if the transmission problem is resolvable, then for any nonempty subset $\gamma \subseteq \mathcal{I}$, any order \prec_γ in γ and any cut $\mathcal{U} \in \mathcal{U}_\gamma$,

$$\sum_{k \in \gamma} H(\mathcal{X}_k) \leq \sum_{(i, j) \in \mathcal{E}_\mathcal{U}: \substack{\mathcal{T}_{ij} \cap \gamma \neq \phi, \\ \mathcal{S}_{ij} \cap \gamma \not\subseteq \gamma(\mathcal{T}_{ij})}} C_{ij}. \quad (17)$$

For a K -pairs transmission network, this new bound is an improvement over the max-flow min-cut bound, i.e.,

$$\sum_{(i, j) \in \mathcal{E}_\mathcal{U}: \substack{\mathcal{T}_{ij} \cap \gamma \neq \phi, \\ \mathcal{S}_{ij} \cap \gamma \not\subseteq \gamma(\mathcal{T}_{ij})}} C_{ij} \leq \sum_{(i, j) \in \mathcal{E}_\mathcal{U}: \substack{\mathcal{T}_{ij} \cap \gamma \neq \phi, \\ \mathcal{S}_{ij} \cap \gamma \neq \phi}} C_{ij}.$$

This result can be seen as follows. Given a set of nodes \mathcal{U} , let γ denote the set of messages whose source lies in \mathcal{U} and whose sink lies outside \mathcal{U} . If the elements of γ are numbered $1, 2, \dots, |\gamma|$, the network sharing bound states

that the combined transmission rate of the messages in γ is bounded above by the combined capacity of the set consisting of all edges from \mathcal{U} to $\mathcal{V} \setminus \mathcal{U}$ which lie on a path from the source of a message in γ to the sink of the same message or a lower-ordered message in γ . However, in this formulation of the bound, if we relax the constraint on “the same message or a lower-ordered message” to “any other message”, we obtain the max-flow min-cut bound.

C. An Extended Network Sharing Outer Bound for Arbitrary Acyclic Multi-source Multi-sink Networks via Sink Decomposition

In a general multi-source multi-sink network, each sink can request messages from multiple sources. However, this problem can be easily solved by *sink decomposition*, i.e., we decompose each sink t_i into $|\mathcal{D}_i|$ copies, each of them has a single source reconstruction demand and has the same set of connections with other nodes as t_i . For example, by sink decomposition, the network in Fig. 1 can be transformed into the network in Fig. 2. Thus, any general multi-source multi-sink network with arbitrary sink demands can be viewed as a multi-source multi-sink network with non-multicast transmission requirements in which each sink decodes only one source.

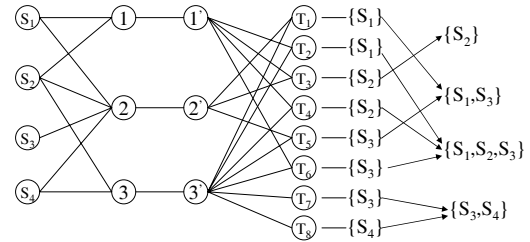


Fig. 2. A sink decomposition for the three-layer network shown in Fig. 1.

By enumerating all possible $|\mathcal{S}|$ -tuples in $\mathcal{T}_1 \times \mathcal{T}_2 \times \dots \times \mathcal{T}_{|\mathcal{S}|}$ where \mathcal{T}_i is the set of sink nodes at which the i th source is to be decoded, we get $\prod_{j \in \{1, \dots, |\mathcal{S}|\}} |\mathcal{T}_j|$ K -pairs transmission subnetworks. Therefore, the intersection of our new outer bounds on the capacity regions with respect to these subnetworks gives an outer bound for the capacity region of the general multi-source multi-sink network.

For any multi-source multi-sink non-multicast network $\mathcal{G}, \forall (j_1, j_2, \dots, j_{|\mathcal{S}|}), j_i \in \mathcal{T}_i$ with non-multicast transmission requirements discussed above, we define $G_{j_1, j_2, \dots, j_{|\mathcal{S}|}}$ as a K -pairs multi-source multi-sink subnetwork of \mathcal{G} by considering only the sinks with the given indices. Let $\mathcal{H}_{j_1, j_2, \dots, j_{|\mathcal{S}|}}$ be the new outer bound on the entropy rates of the sources derived from this subnetwork. The following is a direct consequence of Theorem 2.

Corollary 1: Given a multi-source multi-sink network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with arbitrary transmission requirements \mathcal{D} , if the source transmission problem is resolvable, then

$$\mathcal{H} \subseteq \bigcap_{(j_1, j_2, \dots, j_{|\mathcal{S}|}) \in \prod_i \mathcal{T}_i} \mathcal{H}_{j_1, j_2, \dots, j_{|\mathcal{S}|}}. \quad (18)$$

IV. PROOF OF MAIN RESULTS

A. Proof of Theorem 1:

Let a square matrix with rows and columns indexed by δ be denoted as matrix δ .

- 1) Main channel capacity constraint (13): By observation 1) in III.A., each diagonal tile supplied by the main channel (λ, λ) , $\lambda \cap \gamma \neq \phi$ can be used to tile a matrix $\delta \subseteq \lambda \cap \gamma$, thus contributing a weight $R_\lambda^\lambda(\delta)$ to the total weight of matrix δ . However, the total weight of such contributions from (λ, λ) are constrained by the capacity C_λ^λ .
- 2) Side information channel capacity constraint (14): By observation 2) in III.A., each tile supplied by the side information channel (α, β) can be used to tile a matrix $\delta \subseteq \gamma$ as long as their intersection is nonempty, i.e., $\alpha \cap \delta \neq \phi$, and $\beta \cap \delta \neq \phi$. However, the total weight of such tiles from (α, β) is constrained by the capacity C_α^β .
- 3) Matrix weight constraint (15): By 1), the weight of matrix $\delta \subseteq \gamma$ equals the total weight of diagonal tiles from the main channels (λ, λ) , $\delta \subseteq \lambda$. In order for matrix δ to be tiled, the total weight of rectangular tiles used on any off-diagonal element $(x, y) \in (\delta \times \delta)$ should be no less than its weight.
- 4) Matrix sum weight function (12): The total entropy rate from the sources can be measured by the sum weight of all the tiled matrices. A tiled matrix $\delta \subseteq \gamma$ of unit weight gives a source entropy rate $|\delta|$. \square

B. Proof of Theorem 2:

Suppose a capacity vector $\mathbf{C} = (C_{ij}, (i, j) \in \mathcal{E})$ is admissible for the given network, then for every $\epsilon > 0$, there exists for sufficient large n an $(n, (\eta_{ij}, (i, j) \in \mathcal{E}), (\Delta_l, l \in \mathcal{T}))$ code such that

$$n^{-1} \log \eta_{ij} \leq C_{ij}, \forall (i, j) \in \mathcal{E} \text{ and } \Delta_l \leq \epsilon, \forall l \in \mathcal{T}.$$

Lemma 1: Define inductively,

$$\begin{aligned} U_{ij} &= F_{ij}(\{X_{i'}^n : i' \in \mathcal{S}, (i', i) \in \mathcal{E}\}, \\ &\quad \{U_{i'i} : i' \notin \mathcal{S}, (i', i) \in \mathcal{E}\}) \end{aligned} \quad (19)$$

as the information flow on edge $(i, j) \in \mathcal{E}$. Then we have

$$1) \quad H(U_{ij}) \leq \log \eta_{ij} \leq nC_{ij}, \quad (20)$$

$$\begin{aligned} 2) \quad &H(U_{ij} | \{X_{i'}^n : i' \in \mathcal{S}, (i', i) \in \mathcal{E}\}, \\ &\quad \{U_{i'i} : i' \notin \mathcal{S}, (i', i) \in \mathcal{E}\}) = 0, \end{aligned} \quad (21)$$

- 3) by Fano's inequality, $\exists \delta$ depending on ϵ such that $\delta \rightarrow 0$ as $\epsilon \rightarrow 0$ and for any $l \in \mathcal{T}$,

$$n^{-1} H(X_l^n | \{U_{il} : (i, l) \in \mathcal{E}\}) \leq \delta, \quad (22)$$

- 4) furthermore,

$$H(U_{ij} | \{X_{i'}^n : i' \in \mathcal{S}_{ij}\}) = 0, \quad (23)$$

- 5) for any $l \in \mathcal{T}$ and for any cut $\mathcal{U} \in \mathcal{U}_{\mathcal{T}}$,

$$n^{-1} H(X_l^n | \{U_{ij} : (i, j) \in \mathcal{E}_{\mathcal{U}}\}) \leq \delta. \quad (24)$$

Proof: 1) follows from the definition of the $(n, (\eta_{ij}, (i, j) \in \mathcal{E}), (\Delta_l, l \in \mathcal{T}))$ code. 2) and 3) are consequences of the definitions of the encoders and decoders. And 4) and 5) can be easily derived from 2) and 3) by induction. \square

Let $\gamma = \{i_1 \prec_\gamma i_2 \prec_\gamma \dots \prec_\gamma i_{|\gamma|}\}$. Consider any cut $\mathcal{U} \in \mathcal{U}_\gamma$. For convenience of notation, we define a set

$$\Delta = \{U_{ij} : (i, j) \in \mathcal{E}_{\mathcal{U}}, \mathcal{T}_{ij} \cap \gamma \neq \phi, \mathcal{S}_{ij} \cap \gamma \not\subseteq \gamma(\mathcal{T}_{ij})\},$$

and another set $\Lambda = \{X_k^n : k \notin \gamma\}$.

Thus we have

$$\begin{aligned} &\sum_{(i,j) \in \Delta} nC_{ij} + \sum_{k \notin \gamma} H(X_k^n) \\ &\geq \sum_{(i,j) \in \Delta} H(U_{ij}) + \sum_{k \notin \gamma} H(X_k^n) \\ &\geq H(\Delta, \Lambda) \\ &= H(\Delta, \{X_k^n : k \in \mathcal{S}\}) - H(\{X_k^n : k \in \gamma\} | \Delta, \Lambda) \\ &\stackrel{(a)}{\geq} H(\{X_k^n : k \in \mathcal{S}\}) - n|\gamma|\delta \\ &= \sum_{k \in \gamma} H(X_k^n) + \sum_{k \notin \gamma} H(X_k^n) - n|\gamma|\delta. \end{aligned}$$

Therefore,

$$\sum_{k \in \gamma} n^{-1} H(X_k^n) \leq \sum_{(i,j) \in \Delta} C_{ij} + |\gamma|\delta.$$

Let $\epsilon \rightarrow 0$ and $n \rightarrow \infty$, then $\delta \rightarrow 0$ and we complete the proof, where the key step (a) is proved as follows.

From the chain rule of entropy functions

$$H(\{X_k^n : k \in \gamma\} | \Delta, \Lambda) = \sum_{l=1}^{|\gamma|} H(X_{i_l}^n | X_{i_1}^n, \dots, X_{i_{l-1}}^n, \Delta, \Lambda). \quad (25)$$

Thus we only need to show that

$$\begin{aligned} &H(X_{i_l}^n | X_{i_1}^n, \dots, X_{i_{l-1}}^n, \Delta, \Lambda) \\ &\leq H(X_{i_l}^n | X_{i_1}^n, \dots, X_{i_{l-1}}^n, \{U_{ij} : (i, j) \in \mathcal{E}_{\mathcal{U}}, \\ &\quad i_l \in \mathcal{T}_{ij}, \mathcal{S}_{ij} \cap \gamma \not\subseteq \gamma(\mathcal{T}_{ij})\}, \Lambda) \\ &\stackrel{(b)}{=} H(X_{i_l}^n | X_{i_1}^n, \dots, X_{i_{l-1}}^n, \Lambda, \{U_{ij} : (i, j) \in \mathcal{E}_{\mathcal{U}}, \\ &\quad i_{\mathcal{T}_{ij}} = i_t, i_l \in \mathcal{T}_{ij}, \\ &\quad \mathcal{S}_{ij} \cap \gamma \not\subseteq \{i_1, \dots, i_{t-1}\} : t = 1, \dots, l\}) \\ &\stackrel{(c)}{\leq} H(X_{i_l}^n | \{U_{ij} : (i, j) \in \mathcal{E}_{\mathcal{U}}, i_{\mathcal{T}_{ij}} = i_t, i_l \in \mathcal{T}_{ij}, \\ &\quad \mathcal{S}_{ij} \cap \gamma \subseteq \{i_1, \dots, i_{t-1}\} : t = 1, \dots, l, \\ &\quad \{U_{ij} : (i, j) \in \mathcal{E}_{\mathcal{U}}, i_{\mathcal{T}_{ij}} = i_t, i_l \in \mathcal{T}_{ij}, \\ &\quad \mathcal{S}_{ij} \cap \gamma \not\subseteq \{i_1, \dots, i_{t-1}\} : t = 1, \dots, l\} \\ &\quad : t = 1, \dots, l) \\ &\stackrel{(d)}{=} H(X_{i_l}^n | U_{ij} : (i, j) \in \mathcal{E}_{\mathcal{U}}, i_l \in \mathcal{T}_{ij}) \\ &\leq n\delta. \end{aligned}$$

The noted steps are explained as follows:

(b) $i_l \in \mathcal{T}_{ij}$ implies that $i_{\mathcal{T}_{ij}} \prec_\gamma i_l$.

(c) by fact 2), it follows that

$$H(U_{ij} : (i, j) \in \mathcal{E}_U, \mathcal{S}_{ij} \cap \gamma \\ \subseteq \{i_1, \dots, i_{l-1}\} | X_{i_1}^n, \dots, X_{i_{l-1}}^n, \Lambda) = 0,$$

$$\forall l \in \{1, \dots, |\gamma|\}, \forall \mathcal{T}_{ij} \subseteq \{1, \dots, |\mathcal{T}|\}, \mathcal{T}_{ij} \neq \emptyset \text{ and} \\ H(Y|X) \leq H(Y|g(X)), \text{ for any function } g.$$

(d) similar to (b).

Therefore, Theorem 2 is proved. \square

An intuitive sketch of the above argument is that given all the messages in the complement of γ (i.e., Λ) and given the data transmitted on edges in Δ , one may decode all the messages in γ as follows: first one may decode message 1 (in the given order) because every path from any source to the sink of message 1 either originates in Λ or crosses Δ ; next one may decode message 2 because every path from any source to the sink of message 2 either originates in $\Lambda \cup \{s_1\}$ or crosses Δ , and message 1 has already been decoded. Continuing in this way, one may decode all messages in γ in the order \prec_γ determined by their numbering.

V. AN EXAMPLE

Consider the regular 3-pairs three-layer network in Fig. 3. The max-flow min-cut bound is

$$H(X_i) \leq 3, \forall i \in \{1, 2, 3\}, \\ H(X_i) + H(X_j) \leq 7, \forall i, j \in \{1, 2, 3\}, i \neq j, \\ H(X_1) + H(X_2) + H(X_3) \leq 10.$$

By Theorem 1, it is not difficult to find that the linear programming inner bound is

$$H(X_i) \geq 3, \forall i \in \{1, 2, 3\}, \\ H(X_i) + H(X_j) \geq 5, \forall i, j \in \{1, 2, 3\}, i \neq j, \\ H(X_1) + H(X_2) + H(X_3) \geq 7.$$

An optimal tiling solution for $\gamma = \mathcal{I}$ is given in Fig. 4.

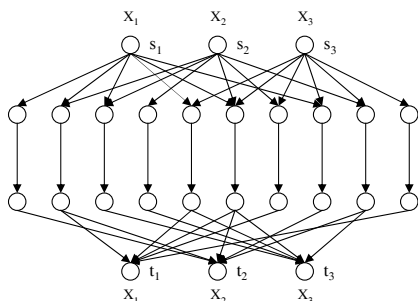


Fig. 3. A regular 3-pairs three-layer network, where $C_{S_{ij}}^{\mathcal{T}_{ij}} = 1$ for all channels $(S_{ij}, \mathcal{T}_{ij})$.

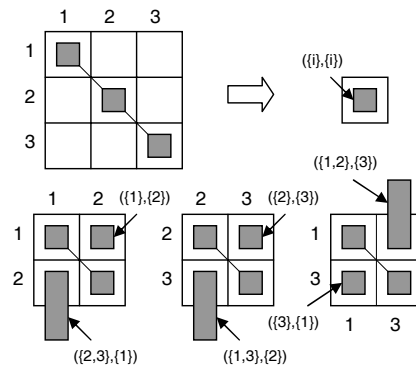


Fig. 4. An optimal tiling solution for the network in Fig. 3., where the diagonal tile $\{1, 2, 3\}$ was used to tile a degraded matrix $\{i, i \in \{1, 2, 3\}\}$.

By Theorem 2, the network sharing outer bound is

$$H(X_i) \leq 3, \forall i \in \{1, 2, 3\}, \\ H(X_i) + H(X_j) \leq 5, \forall i, j \in \{1, 2, 3\}, i \neq j, \\ H(X_1) + H(X_2) + H(X_3) \leq 7.$$

For example, the outer bound for $\gamma = \mathcal{I}$ can be achieved with the order $\gamma = \{1 \prec 2 \prec 3\}$, that is

$$H(X_1) + H(X_2) + H(X_3) \\ \leq C_{12}^{12} + C_{13}^{13} + C_{123}^{123} + C_{23}^1 + C_{13}^2 + C_{23}^{23} + C_3^1 = 7.$$

Therefore, the inner and outer bounds are tight in this example.

VI. CONCLUSION

In this paper, we proved a linear programming inner bound for regular K -pairs acyclic three-layer networks and a network sharing bound for general acyclic multi-source multi-sink networks. The techniques used in the proofs reveal some of the basic mechanisms of multi-source multi-sink network coding.

Although the network sharing outer bound provides significant improvements over the classic max-flow min-cut bound, it is shown still not tight in general [2]. However, it is still an open question whether the proposed linear programming inner bound is tight.

A sink decomposition technique is used in this paper when we derived the outer bound for networks with arbitrary transmission requirements. Since this technique gives a tight result in the single source multicast case, it will be interesting to know whether it still causes no losses in more general cases.

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