Credit Risk in a Network Economy

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Abstract

We develop a structural model of credit risk in a network economy, where any firm can lend to any other firm, so that each firm is subject to counterparty risk either from direct borrowers or from remote firms in the network. This model takes into account the role of each firm's cash management. We show that we can obtain a semi-closed form formula for the price of debt and equity when cash accounts are buffers to bankruptcy risk. As in other structural models, the strategic bankruptcy decision of shareholders drives credit spreads, and differentiates debt from equity. Cash flow risk also causes credit risk interdependencies between firms. Our model applies to the case where not only financial flows but also operations are dependent across firms. We use queueing theory to obtain our semi-closed form formulae in steady state. We perform a simplified implementation of our model to the US automotive industry and show how we infer the impact on a supplier's credit spreads of revenue changes in a manufacturer or even in a large car dealer.

(Credit Risk; Contagion; Queueing Networks)

1 Introduction

We motivate our model by an example in the US automotive industry. Consider a major part supplier, such as Visteon. The main goal of our model is to price the debt issued by companies such as Visteon, whose credit standing depends on the credit standing of its counterparties. The credit standing of Visteon depends critically on the credit standing of major US car manufacturers, such as Ford and GM, to which Visteon extends trade credit. In turn, Ford and GM's credit standing depends on the credit standing of their borrowers. Car dealers such as Autonation, United Technologies, and Lithia [1] typically borrow from car manufacturers, in the form of trade credit. Therefore, the credit spread of Visteon should depend not only on the credit spread of Ford and GM, but also, indirectly, on the credit spread of, say, Autonation. Moreover, in this *network economy*, one should expect that the fluctuations in Visteon's debt price should be somewhat proportional to the fluctuations in Autonation's revenue, and the proportion ratio should be related to the (indirect) exposure of Visteon to Autonation.

Modelling credit risk dependencies between firms, and hence direct contagion effects, is thus the principal motivation of this article. We address this issue in the context of structural credit risk models, which have been initiated by Merton (1974). These models [2] tend however to consider firms in isolation from the network of economic relationships they are embedded in. As we shall see, our model thus extends structural credit risk models to a network economy, under a particular set of assumptions.

Structural credit risk models typically capture the following economic features: variability of asset value or revenue, contagion of default from borrowers to lenders, strategic bankruptcy of the shareholder, cash-flow

[1] A network representing a simplified US automotive industry is graphed in figure 2.
[2] We delay a more extensive discussion of the literature to the end of this section.

risk, tax benefit of debt and optimal capital structure, bankruptcy costs and recovery rates, term structure of interest rates, and dividend/reinvestment policy. We do not model capital structure, term structure of interest rates, nor bankruptcy costs. While constant interest rates and recovery rates are usual simplifications in a first stage of modelling, we stress that our model is limited to a network of firms that have a fixed, exogenously determined, level of debt. Moreover, not only the quantity of debt is fixed, but also the network of lending and borrowing is fixed, that is, firms have preferred lenders, and the amount borrowed from them does not change with time.

This is the first assumption of our model: *(A1) the network structure of debt payments is fixed*. Even when this assumption is made, it is not clear how to incorporate these interdependencies in a structural framework. The main challenge is the following. In a (single-firm) structural model a firm declares bankruptcy when asset values fall under a (generally) fixed threshold, related to the amount of debt borrowed. In the strategic bankruptcy type of models, initiated by Leland (1994), shareholders select this bankruptcy threshold so as to maximize the value of equity. When many, say I firms, are involved in a network, it is a formidable task to determine what the bankruptcy threshold should be. One can imagine the complexity of determining I different optimal stopping rules based on I dependent processes, that is, I different bankruptcy rules based on I different asset value processes in a Nash-type equilibrium. Two key observations guided us around this issue:

- rather than modelling stock variables (like assets), it is easier to think in terms of flows (that is, revenue and costs)
- the inextricable interdependence across firms can be reduced if cash flow risk is treated differently from endogenous bankruptcy risk.

We call *cash flow risk* the risk of either early or delayed debt payments that do not result in the shareholders losing ownership of the company, as opposed to bankruptcy risk. In our model, bankruptcy is a decision driven by shareholders. Upon bankruptcy, shareholders lose their stake in the company, i.e., they stop receiving dividends. Bankruptcy costs are then incurred by the debtholders. Bankruptcy typically occurs when earnings are significantly and durably lower than debt payments. This bankruptcy option is thus an asset for shareholders and a liability for debtholders. Cash flow risk has a more ambiguous effect though, as neither shareholders nor debtholders desire to enter costly bankruptcies because of a temporary shortfall of liquidities. A firm's cash account thus serves as a buffer, and mitigates cash flow risk.

Our set-up allows us to use queueing theory as a particularly well-suited technical tool. We exploit the quasi-independence property that can occur in queueing networks, and show that the firms strategic bankruptcy policies become quasi-independent, which leads us to price equity and debt in steady state for the case of firms that hold "sufficiently" large cash accounts. The formula is in semi-closed form, that is, only the endogenous bankruptcy risk component is in closed form. It is a function of the following parameters: debt coupon rate and principal, and parameters of the payout and revenue. In the case of large cash accounts, counterparty risk and contagion will be due solely to cash flow risk, and cash flow risk can be either of macroeconomic or idiosyncratic type. Nevertheless, equity and debt follow different dynamics, because of the endogenous bankruptcy component. Our approach also allows to quantify when decoupling cash flow risk from bankruptcy risk is justified[3].

The model applies to both trade debt and financial debt since it is a cash flow based model. This is a helpful guide for the scope of applicability of the model. Counterparty credit risk is a particularly important concern in industries where outsourcing is important as external capital markets have substituted to internal ones, thus increasing exposure. Also, as mentioned earlier, our model is most relevant for firms with sizeable cash accounts and rigid debt structure. A typical application of our model is therefore trade credit between large manufacturing firms, as discussed earlier in the context of the US automotive industry. A second typical application is debt in countries such as Germany and Switzerland, where banks and not capital markets are the main sources of funds, and banks are conservative. A third application is sovereign debt, where the identity of major lenders does not change substantially over time.

[3] Even if there is no formula for that, we can quantify this by simulation.

We now turn to the related literature. As mentioned earlier, our approach is an extension of the structural models of credit risk. Structural form models are not necessarily very good at fitting data. For example, Eom, Helwege and Huang (2004) confirms that Merton's (1974) model tends to underestimate spreads while other structural models (such as Longstaff and Schwartz (1995) and Leland and Toft (1996)) tend to overestimate spreads in higher risk firms. Nonetheless, structural models provide economic insights about the economic drivers of the spreads. This gives such models value beyond their fitting ability. Anderson and Sundaresan (2000) also document that (single-firms) structural models underestimate credit spreads, but less so when bankruptcy is strategic. The mathematical difficulty of generalizing strategic bankruptcy to many firms has been alluded to in Giesecke (2002) p.15. Mella-Barral and Perraudin (1997) and Goldstein, Ju, and Leland (2001) pioneered the description of credit risk in terms of flows. Anderson and Sundaresan (1996) argue that shareholders wish to avoid bankruptcies driven by lack of liquidities. A popular alternative to our model, within the family of structural models, is copula-based models (see Hull and White(2000), Frey and McNeill (2001) and Giesecke (2005)). Copula-based models also extend single-firm models to multiple firms. In these models dependency between defaults derives from the dependency between equity prices. These models do generally not offer a theoretical justification of this crucial link; practitioners usually calibrate them by optimizing the fit between observed credit spreads and model-generated credit spreads. In contrast, we explain this link by the flows of revenues, costs, and payments between the different firms.

There are clearly other approaches to model credit risk interdependencies than structural models. While classical reduced-form models (e.g., Duffie and Singleton (1999) or Jarrow and Turnbull (1995)) do not address credit risk contagion, more recent models have attempted to address the issue of counterparty risk in a simpler setting: what happens to a company's credit risk if the default process is conditional on another company's credit situation. Jarrow and Yu (2001) solve this problem when the relationship is unidirectional: company A's credit risk is impacted by company B's credit risk, but B's credit risk is not impacted by A's. Their model can be generalized to looping effects, but with the loss of analytical tractability. A more tractable generalization to looping effects between two counterparties has been given by Collin-Dufresne, Goldstein and Hugonnier (2004). These looping or "feedback" effects are important, as showed among others by Egloff, Leippold, and Vanini (2004). We stress that our model handles these looping effects seamlessly. Davis and Lo (2001), Giesecke and Weber (2004), Frey and Backhaus (2003), Neu and Kuehn (2004), Egloff et al (2004) study the dynamics of default in a network of firms subject to both macro and microeconomic (i.e., counterparty-related) risks. These models give an explanation of the kurtosis of the distribution of loss given default, which is observed to be higher than what typical models would predict without counterparty risk. When the number of firms becomes large, network effects are dominated by macroeconomic effects (Frey and Backhaus (2003)). Giesecke and Weber (2004) use the theory of interacting particle systems to show among others that counterparty risk is inversely proportional to the degree of connectedness of the network of counterparties. This effect is less pronounced when macroeconomic risk is large. Like Giesecke and Weber, we focus on steady state. Our stylized examples reach the same conclusions about diversification as theirs, however we have not attempted yet to characterize these features analytically. Our focus is indeed different; whereas the aforementioned authors study the distributional properties of default in a network, we focus on debt pricing.

Our contributions are thus the following. First, as mentioned above, we develop a model of debt and credit spread pricing subject to a network of counterparty risks that is amenable to a semi-closed form formula, because quasi-independence between each firm strategic bankruptcy policy occurs when cash accounts serve as buffers. In particular, we obtain the impact of real economic interdependencies on credit spreads. Second, our use of queueing theory to solve this type of finance problem seems to be a novelty; indeed one of the queueing theory results that we expose (proposition 3) is a new result, necessitated by the character of this particular "application" of queueing theory. Third, we briefly show some applications of the theory, such as discussing the calibration of our model to a manufacturing example, the US automotive industry, and studying which type of counterparty network is more robust against contagion.

2 Model

The uncertainty is described by the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, Q)$. We call X the vector of state variables that are public knowledge, and summarize in assumption (A8) what this vector is composed of. X generates the filtration $\{\mathcal{F}_t^X\}$, which is a subfiltration of $\{\mathcal{F}_t\}$. While $\{\mathcal{F}_t^X\}$ describes the information available to public investors, $\{\mathcal{F}_t\}$ describes the information available to firm management. In this situation of incomplete information (as in Giesecke (2004) and Collin-Dufresne, Goldstein, and Helwege (2004)), managers are not allowed to trade in the public market, for otherwise trades would reveal inside firm information. The risk-neutral measure Q is an equivalent measure under which asset prices, when discounted at constant interest rate r, are martingales in the filtration $\mathcal{F}_t^X$. In other terms, the market value at time t of a claim paying V_T at time T is given by:

$$V_t = \exp(-r(T-t))E^Q[V_T|\mathcal{F}_t^X] \tag{1}$$

A second measure N will be introduced in section 2.2. We will denote by E^T and Var^T expectation and variance under the T measure, where $T \in \{Q, N\}$. Statements about independence of random variables refer implicitly to the Q measure, unless specified.

2.1 Financial Assumptions

We model the *financial unit*, or *cash management unit*, of each firm i, with $i = 1..I$. This financial unit receives (net) operational revenue R_i from the production unit, and redistributes C_i to the production unit as "re-investment in production". With this definition, revenue minus costs equals EBIT. We stress that the model will be more realistic if both R_i and C_i are taken net of "fixed production costs", which emanate from long-term strategic decisions of the firm, and should not be submitted to the dictates of the financing policy. C_i will thus represent "above-average production costs". The financial unit of the firm also receives financial and trade debt payments P_{ji} from firm j (with $j \neq i$), pays debt P_{ij} to firm j (with $j \neq i$), and dividends D_i to shareholders. To balance revenue and expenses, the firm maintains a cash account. The equations of the cash account Z are thus:

$$Z_i(t) - Z_i(0) = A_i(t) - H_i(t) \tag{2}$$

where A_i is the total revenue, H_i the (total) expenses, given by:

$$A_i = R_i + \sum_{j \neq i} P_{ji} \tag{3}$$

$$H_i = C_i + D_i + \sum_{j \neq i} P_{ij} \tag{4}$$

Revenue R, debt payments P, and non-debt expenses $C_i + D_i$ are increasing processes. An example of a network is given in figure 1. The financial assumptions of the model are the following.

(A1) The network structure of debt payments is fixed.

For simplicity in the sequel of the text, debt payments consist of interest plus principal reimbursement, minus negative amortization[4]. The principal on the debt issued by firm i to firm j is thus a fixed amount, say L_{ij}, and the coupon rate y_i is also fixed. By convention, $L_{ii} = 0$.

(A2) *Debt has infinite maturity. Interest is paid continuously.*

[4] Our model can also handle lines of credit, where the principal varies with time. Positive changes in proceeds are immediately absorbed by the production unit, while negative changes reflect the sale or mortgaging of a production asset to reduce debt. However the proportion of principal owed to each lender must remain fixed.

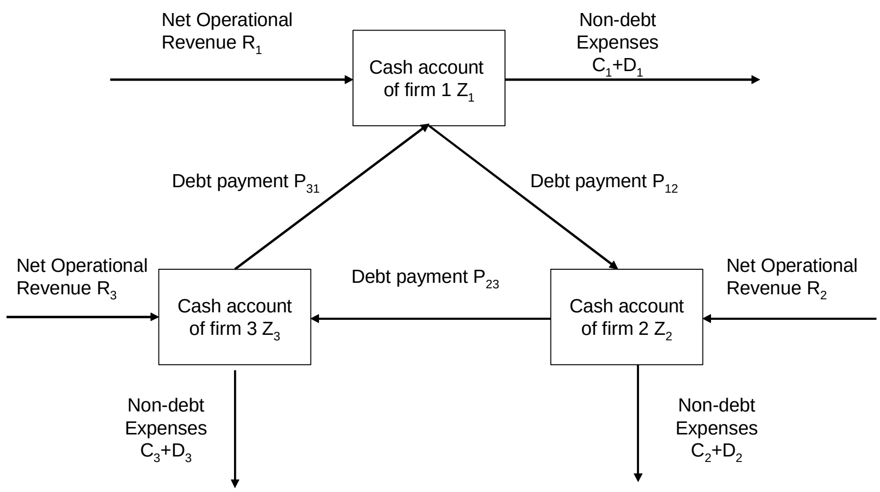

Figure 1: A network of 3 firms

Infinite maturity is a standard assumption (see e.g., Leland (1994)), used to facilitate analysis. In our context, we will use steady-state results from queueing theory, and infinite maturity is therefore appropriate.

(A3) *The cash account serves as a buffer and reduces the probability of bankruptcy.*

When total revenue is low, firms spend less, but the effect is mitigated by the cash account, which acts as a buffer against contagion. Conversely, when total revenue A_i is high, firms increase their cash account Z_i but also spend more to reimburse missed debt payments. The cash account smooths revenue to stabilize debt payments, that is:

$$Var^Q[\frac{P_{ij}(t)}{P_{ij}(0)}] << Var^Q[\frac{R_i(t)}{R_i(0)}] \tag{5}$$

Also, all debt ends up being paid, as firms are refinanced at each default time (see (A6)) to compensate for loss given default. That is:

$$\lim_{t\to\infty} \frac{P_{ij}(t)}{t} = y_i L_{ij} \tag{6}$$

The randomness of the mechanics of the cash management unit, which from now we view as a "black box", reflects the randomness of the environment facing the cash management unit: difficulties in extending lines of credit, reducing costs, or modifying dividends. What are realistic dynamics of this black box? The finance empirical literature shows that cash accounts are mean-reverting, and this has been rationalized by Opler et al (1999), Cossin and Hricko (2002), and Mello and Parsons (2000). We will show later how we model the cash management unit as a queue, resulting in a mean-reverting cash account and debt payments with "low" variability, i.e., that respect (5).

(A4) *Firms maximize the value of equity by strategically declaring bankruptcy.*

Shareholders of firm i will default at default time T_i^k so as to maximize the value of their equity[5]. A priori they condition this decision on all the public information available. This endogenous bankruptcy setting is similar to Leland's (1994), where shareholders default when they foresee a period of significantly negative dividends ahead.

(A5) *The payout to investors is an exogenously determined proportion of total expenses.*

[5] As we shall see in assumption (A6) a firm can default several times, so that T_i^k is the k-th default time of firm i.

As mentioned in the introduction, we do not net out operational revenue R_i with above-average production costs C_i. This allows to capture the "lag" between re-investment in production and its effect on revenue. This assumption also leaves us enough flexibility to model two cases: (i) a firm where the dividend policy is more important than the reinvestment policy, and (ii) the reverse. We favor the second interpretation, that is, the firm is production-driven, and the optimal re-investment policy is continuously assessed, relative to the size of the firm. We call δ_i the exogenously determined payout ratio. Payout, i.e., payments to debtholders and shareholders are proportional to the payout ratio times total expenses[6]:

$$\lim_{t\to\infty} \frac{D_i(t) + \sum_{j\neq i} P_{ij}(t)}{t} = \lim_{t\to\infty} \frac{\int_0^t \delta_i(s) dH_i(s)}{t} \tag{7}$$

whereas reinvestment follows:

$$\lim_{t\to\infty} \frac{C_i(t)}{t} = \lim_{t\to\infty} \frac{\int_0^t (1-\delta_i(s)) dH_i(s)}{t} \tag{8}$$

The payout rate jumps at all bankruptcy times (see assumption (A6)), but otherwise follows geometric Brownian motion, as in Goldstein et al (2001) :

$$\frac{d\delta_i(t)}{\delta_i(t)} = \mu_i dt + \sigma_i dW_i^{\delta}(t) \qquad t \in [0, T_i^1) \tag{9}$$

where W_i^{δ} is a $\mathcal{F}_t^X-$Brownian motion in the Q measure. Negative "above-average costs" correspond to the case where fixed costs happened to be less than forecast, and some extra capital is returned to investors. Similarly, negative dividends correspond to a recapitalization of the firm by its shareholders, as in Goldstein et al (2001). We do not assume any specific independence between the payout ratio of different firms. We can therefore model macroeconomic risk as well as idiosyncratic risk.

(A6) *Upon default times T_i^k, bankruptcy costs are incurred by the debtholders, who, alongside with new shareholders, inject an extra quantity of capital to restore the firm to its normal level of efficiency.*

In our model, firms are not liquidated upon default but are refinanced. At each default, equity is issued to new shareholders. The capital injected results in a jump of the payout ratio at time T_i^k. Without loss of generality the "normal level of efficiency" is the one prevailing at time zero. The payout ratio δ_i has therefore a regenerative structure, with:

$$0 < \delta_i(T_i^k) = \delta_i(0)$$

and follows geometric Brownian motion as in (9) between two successive bankruptcy times. The bankruptcy cost BC_i^k incurred at that time T_i^k is equal to w_i times the present value of all future revenue of the firm, with $0 \leq w_i \leq 1$ so that recovery rates are $1 - w_i$.

(A7) Cash accounts are not appropriated upon default.

Protective covenants prevent shareholders from appropriating the cash account upon default. Also, since debtholders keep their stake in the company for ever, they do not need to appropriate the cash account.

(A8) The operational revenue and the value of the cash account of each firm is its private knowledge.

Whereas market participants can observe expenses, and how they are allocated between debt payments, dividends and reinvestment, the cash account is $\mathcal{F}$-adapted but not $\mathcal{F}^X$-adapted. For this to occur, equations (2) and (3) also require that revenue be not observable. We argue that firms protect themselves from adverse "panic" reactions of investors by not disclosing their cash account fully. Beyond the practical difficulties of presenting, say daily accounting reports instead of quarterly ones, there is a clear informational advantage in presenting only "smoothed" results. However, incompleteness of information does not prevent market

[6] We explain in the next section the exact dynamics; assumption (A6) implies that firms survive all the way until ∞.

participants from trying to estimate the parameters of R_i and of Z_i. The vector X of public information is thus:

$$X = \begin{bmatrix} C \\ D \\ \delta \\ P \\ M \end{bmatrix} \tag{10}$$

where M is a vector of macroeconomic variables[7].

(A9) The optimal default policy is sufficiently regular.

This assumption, of technical nature, allows for an approximation in which the bankruptcy decision of firm i is independent from contemporaneous debt payments from firm j, when (5) holds. This assumption is fully elaborated in the appendix.

2.2 Representation of the Model as a Queueing Network

Rather than giving relations between R_i, Z_i, H_i, P_{ij} and $C_i + D_i$, our queueing model specifies relationships between $R_i^{(n)}, Z_i^{(n)}, H_i^{(n)}, P_{ij}^{(n)}$, and $C_i^{(n)} + D_i^{(n)}$, with:

$$R_i = \frac{R_i^{(n)}}{n}; \quad Z_i = \frac{Z_i^{(n)}}{n}; \quad H_i = \frac{H_i^{(n)}}{n}; \quad P_{ij} = \frac{P_{ij}^{(n)}}{n}; \quad C_i + D_i = \frac{C_i^{(n)} + D_i^{(n)}}{n} \tag{11}$$

Since, as we shall see, the size of each increment of $R_i^{(n)}$ is equal to one, the size of each cash arrival R_i is equal to $\frac{1}{n}$.Our results will hold asymptotically for large values of n. When n is large, a diffusion approximation (proposition 2) will show that the value of the cash account Z_i is approximately mean-reverting, as required by assumption (A3).

2.2.1 Net Operational Revenue

The process $R_i^{(n)}$ is adapted to the filtration of firm management $\mathcal{F}$ but not adapted to the filtration of public investors $\mathcal{F}^X$ (see assumption (A8)). It is a Cox, or doubly-stochastic Poisson process (see e.g., Brémaud (1981)[8]). A Cox process is defined by another process, namely the *intensity*: given the realization of the intensity, a Cox process is a non-stationary Poisson process with rate at time t equal to the time-average of the intensity up to t [9]. We write $\nu_i^{(n)}$ for the intensity of $R_i^{(n)}$. The process $\nu_i^{(n)}$ is proportional to the product of a square root process k_i oscillating around one and a $\mathcal{F}_\infty-$measurable (but not $\mathcal{F}_t^X$-measurable) random variable $\bar{\nu}_i$ representing the long-run value of the intensity. Thus,

$$\begin{aligned} \nu_i^{(n)}(t) &= n\bar{\nu}_i k_i(t) && (12) \\ dk_i &= \frac{1}{m_i}(1-k_i)dt + \frac{m_i}{\sqrt{\bar{\nu}_i}}\sqrt{k_i}dW_i^k \qquad t \le T^f && (13) \\ k_i(0) &= 1 && (14) \\ k_i(t) &= 1 \qquad t \ge T^f && (15) \end{aligned}$$

where W_i^k is a $\mathcal{F}_t-$Brownian motion in the Q- measure. The inverse of the speed of mean-reversion m_i is a constant. We do not impose independence between W_i^k and W_j^k, nor between $\bar{\nu}_i$ and $\bar{\nu}_j$. As a result,

[7] A consequence of this assumption is that cash flow risk will appear even in steady-state, as can be seen from (27) to (29). If the steady-state value of revenue per unit of time were known with certainty, the model would only exhibit endogenous bankruptcy risk.

[8] We depart slightly from Brémaud's definition of the filtrations. In our definition $\nu_i^{(n)}(t)$ is $\mathcal{F}_t$ - measurable but not $\mathcal{F}_0$ -measurable.

[9] We chose a Cox process rather than a Poisson process for revenue in order to implement (5).

we can model both macroeconomic and idiosyncratic risks. The *final time* T^f is a time that is sufficiently high so that the value of discounted cash flows occurring thereafter has a negligible impact on pricing [10]. We introduce a *reference measure* N where the intensity of $R_i^{(n)}$ is given by $n\bar{\nu}_i$. For convenience, the payout rate δ is assumed independent from $\bar{\nu}$ in the reference measure.

2.2.2 Network Model

For technical reasons (see, e.g., Gross and Harris (1985)), rather than splitting units of incoming of revenue to various claimants, it is preferable to randomly assign a full unit of revenue to only one claimant, in such a way that, on average, revenue is adequately distributed. Since the random variables governing routing are assumed independent from all other variables (in both measures), this so-called *Bernoulli routing* is completely innocuous statistically. The routing parameters λ_{ij} represent then both the probability that an unit of expenses from firm i will be fed back into firm j and the following proportion:

$$\lambda_{ij} = \frac{\text{debt payments from firm } i \text{ to firm } j \text{ over a year}}{\text{average annual expenses of firm } i} \tag{16}$$

$$= \lim_{t\to\infty} \frac{y_i L_{ij} t}{H_i(t)} \tag{17}$$

For λ_{ij} to be a proportion, we need:

$$0 \leq \sum_j \lambda_{ij} \leq 1 \tag{18}$$

We do not consider firms that do not fulfill (18), that is, firms that plan to systematically default on their debt. We enhance our model by allowing non-debt expenses to be more variable than debt payments (see footnote 11). Thus, we set for each firm two cash accounts, Z_i^A and Z_i^B. The total value of the cash account is then $Z_i = Z_i^A + Z_i^B$, and (2) holds. A proportion $\sum_{j\neq i} \lambda_{ij}$ of revenue goes by Bernoulli routing to account Z_i^A, while the remainder goes to Z_i^B. The cash from Z_i^A serves to pay debt, while the cash from Z_i^B serves to pay non-debt expenses (see figure 1 in the companion paper, Schellhorn(2004)). As a result,

$$\lim_{t\to\infty} \frac{C_i(t) + D_i(t)}{t} = (1 - \sum_{j\neq i} \lambda_{ij}) \lim_{t\to\infty} \frac{H_i(t)}{t} \tag{19}$$

In compliance with (6), (7), (17), and (19), the differential of dividends satisfies thus:

$$E^Q[dD_i(t)|\mathcal{F}_t] = \frac{\delta_i(t) - \sum_{j\neq i} \lambda_{ij}}{1 - \sum_{j\neq i} \lambda_{ij}} E^Q[d(C_i(t) + D_i(t))|\mathcal{F}_t] \tag{20}$$

2.2.3 Dynamics of the Cash Account

Conditional on $\bar{\nu}_i$, the cash transit distribution is exponential in the reference measure N. When $\Delta t \to 0$:

$$N(Z_i^{A(n)}(t + \Delta t) = z + 1 | Z_i^{A(n)}(t) = z, \bar{\nu}_i) = \alpha_i \sum_{j\neq i} \lambda_{ij} \Delta t + o(\Delta t) \tag{21}$$

$$N(Z_i^{A(n)}(t + \Delta t) = z - 1 | Z_i^{A(n)}(t) = z, \bar{\nu}_i) = z p_i^A \Delta t + o(\Delta t) \tag{22}$$

[10] This final time is introduced for technical resons, namely existence of the Radon-Nikodym derivative dQ/dR (see the companion paper.)

and all other probabilities are zero. The expected total revenue rate (net revenue plus debt payments) is defined in (23). Same relationships hold for the cash account Z^B, replacing A by B, and with $1-\sum_{j\neq i}\lambda_{ij}$ replacing $\sum_{j\neq i}\lambda_{ij}$ in (21). Proposition 2 shows that the constants p_i^A and p_i^B can be interpreted as the *speed of cash transit*[11]. As can be seen from the conditions of proposition 3, a low speed of cash transit guarantees that assumption (A3) is satisfied .

In the language of queueing theory, our economy will thus consist of a network of Cox/M/∞ queues with Bernoulli routing. Conditionally on the value of $\bar{\nu}$, the economy would be a Jackson network of queues in the reference measure. The next table shows the correspondence between the traditional queueing terminology and our interpretation.

	Queueing Terminology	Financial Interpretation
R_i	Arrival from outside the system	(Net) operational revenue of firm i
P_{ij}	Transit process	Debt payment from firm i to firm j
A_i	Arrival process	Total revenue of firm i
H_i	Departure process	(Total) expenses of firm i
$C_i + D_i$	Exit process	Non-debt expenses of firm i
Z_i	Number of persons in queue	Value of the cash account of firm i
ν_i	Outside arrival rate	(Net) operational revenue rate of firm i
$\bar{\nu}_i$	Outside arrival long-term rate	(Net) operational revenue long-term rate of firm i
α_i	Expected arrival rate	Expected total revenue rate of firm i
p_i	Service rate	Speed of cash transit of firm i

2.2.4 Summary of Results from Queueing Theory

One of the advantages of Jackson networks is that their steady state (or equilibrium) distribution is very tractable. We group in proposition 1 the properties we will need later. They are standard results (see e.g., Gross and Harris (1985) equation 2.66, theorem 2.4 and corollary 2.6 in Kelly (1978)).

PROPOSITION 1. *Suppose steady state is reached in a Jackson network. Let α solve*

$$\alpha_i = \nu_i + \sum_{k\neq i}\alpha_k\lambda_{ki} \tag{23}$$

Then,

$$\bar{Z}_i \equiv E^Q[Z_i] = \frac{\alpha_i}{p_i} \tag{24}$$

The exit processes $C_i + D_i$ are independent Poisson processes with intensity $\alpha_i(1-\sum_{k\neq i}\lambda_{ik})$.

Propositions 2 and 3 are described in full details in the appendix.

Proposition 2 shows that for finite n, the cash accounts of a Jackson network are approximately a (vector) Ornstein-Uhlenbeck process, thereby resembling the mean-reverting behaviour prescribed in assumption (A3).

Proposition 3 shows that it is possible in a network of Cox/M/∞ queues with Bernoulli routing to implement (5), that is, to reduce the variability of debt payments. More specifically, the conditional variance of debt payments given $\bar{\nu}$ can then be any small (but positive) fraction of (i) the conditional variance of revenue and of (ii) the conditional variance of non-debt expenses. This is either a new or an independently rediscovered result. Indeed, traditional applications of queueing theory are focussed more on regularizing the buffers, which are the bottlenecks, than the flows on the networks. A result like proposition 3 would be therefore of little interest in applied queueing theory. For us it is the crucial ingredient that makes queueing theory a realistic tool to describe the money flows in our network of firms.

[11] From proposition 2, one sees also that a low speed of cash transit results in a large cash account. This, taking $p_i^A << p_i^B$ reduces the overall size of the cash account.

3 Main Result

Assumption (A8) means that the value of the claims on a firm does not depend on the contemporaneous value of operational revenue, which is not observable. In steady-state, the value of these claims will depend on the long run operational revenue rate $\bar{\nu}_i$, which the market can estimate given the value of the state variables X . We call $\hat{\nu}$ the market estimator of the long-run operational revenue rate, i.e.:

$$\hat{\nu}_i(t) = E^Q[\bar{\nu}_i|\mathcal{F}_t^X] \tag{25}$$

Since total revenue equals total expenses in steady state, the estimator $\hat{\alpha}$ of the long-run total revenue rate solves:

$$\hat{\alpha}_i = \hat{\nu}_i + \sum_{k\neq i} \hat{\alpha}_k \lambda_{ki} \tag{26}$$

THEOREM *In steady state, the value of equity S, total bankruptcy costs TBC, and debt F are, for finite n, and t not a bankruptcy time:*

$$S_i(t) = \hat{\alpha}_i(t)[\frac{\delta_i(t)}{r-\mu_i} - \frac{\sum_{j\neq i}\lambda_{ij}}{r} + (\frac{\sum_{j\neq i}\lambda_{ij}}{r} - K_i)(\frac{K_i(r-\mu_i)}{\delta_i(t)})^{x_i}] + O(\frac{1}{\sqrt{n}}) \tag{27}$$

$$TBC_i(t) = \hat{\alpha}_i(t)\frac{w_i}{r}\left[\frac{\delta_i(0)}{\delta_i(t)}\right]^{x_i} \frac{[K_i(r-\mu_i)]^{x_i}}{\delta_i(0)^{x_i} - [K_i(r-\mu_i)]^{x_i}} + O(\frac{1}{\sqrt{n}}) \tag{28}$$

$$F_i(t) = \hat{\alpha}_i(t)\frac{\sum_{j\neq i}\lambda_{ij}}{r} - TBC_i(t) + O(\frac{1}{\sqrt{n}}) \tag{29}$$

where:

$$x_i = \frac{1}{\sigma_i^2}[\mu_i - \frac{\sigma_i^2}{2} + \sqrt{(\mu_i - \frac{\sigma_i^2}{2})^2 + 2r\sigma_i^2}] \tag{30}$$

$$K_i = \frac{x_i}{x_i+1}\frac{\sum_{j\neq i}\lambda_{ij}}{r} \tag{31}$$

SKETCH OF PROOF: In full generality, the optimal bankruptcy policy of firm i would depend on the whole vector X. In our model firm i cannot base its bankruptcy decision on:

- debt payments of firm j, because of its lack of variability (proposition 3) and of (A3)
- non-debt expenses $C_j + D_j$ of firm j because of its quasi-independence with $C_i + D_i$
- payout rate δ_j of firm j, because it is not linked to any input of the model.

We use a change of measure to the measure N, so that our network transforms into a Jackson network, and we apply proposition 1. As a result, the strategic bankruptcy decision of each firm is decoupled when n is very large, that is, when the cash accounts are large and absorb most of the cash flow risk. Standard single firm results give the optimal bankruptcy policy. The full proof is given in the appendix. ¥

Let us assume for simplicity that the learning process (25) results in $\hat{\nu}$ being a diffusion, and that $O(\frac{1}{\sqrt{n}})$ is negligible. We apply Ito's lemma to (27) and (29):

$$\frac{dS_i(t)}{S_i(t)} = rdt + \sigma_i^{\delta,S}(t)dW_i^{\delta}(t) + \sigma_i^{\hat{\alpha}}(t)dW_i^{\hat{\alpha}}(t) \tag{32}$$

$$\frac{dF_i(t)}{F_i(t)} = rdt + \sigma_i^{\delta,F}(t)dW_i^{\delta}(t) + \sigma_i^{\hat{\alpha}}(t)dW_i^{\hat{\alpha}}(t) \tag{33}$$

where $\sigma_i^{\hat{\alpha}}$ is the volatility of revenue risk, $\sigma_i^{\delta,S}$ and $\sigma_i^{\delta,F}$ can be calculated from (27) and (29), and $W_i^{\hat{\alpha}}$ is $\mathcal{F}_t^X$-Brownian motion in the risk-neutral measure. The process $W^{\hat{\alpha}}$ can be decomposed into both macroeconomic and idiosyncratic factors, like in other contagion models. Therefore our model will capture the effect of contagion through both macroeconomic and firm-specific risks. It suffices that firms be indirectly linked in the counterparty network for contagion to occur, as in the model by Kodres and Plisker (2002) p. 785. This theorem leads us to the following conclusions:

- cash flow risk is decoupled from strategic bankruptcy risk for both debt and equity
- the magnitude of cash flow risk is the same for both debt and equity; what differentiates them is strategic bankruptcy risk
- the dynamics of default are a generalization of default dynamics from the single-firm setting; they are described in the next subsection.

Interdependencies affect credit spreads via delayed payments. Shareholders will normally declare bankruptcy only exceptionally, when all alternatives have been tried, resulting in a much longer time-to-default. The following table describes which type of risk is present in our model, as a function of two parameters of a firm, size of the cash account, and volatility of revenue risk $\sigma_i^{\hat{\alpha}}$. The theorem does not hold when revenue is highly uncertain and the cash account is small.

	Low revenue volatility	High revenue volatility
Large cash account	bankruptcy risk	cash flow + bankruptcy risk (decoupled)
Small cash account	bankruptcy risk	not modelled

Credit derivatives could also be priced according to this framework, and not only debt. However, as we mention in the introduction, structural models in general do not fit data very well, and reduced-form models are maybe more appropriate for pricing credit derivatives.

3.1 Dynamics of Default

When the term $O(\frac{1}{\sqrt{n}})$ is sufficiently small in (27) to (29), the risk neutral dynamics of default are approximately:

$$\begin{aligned} &Q(\text{firm 1 defaults before } t,\ldots,\text{firm } I \text{ defaults before } t) = \\ &Q(\delta_1(s) \leq K_1(r-\mu_1),\ldots,\delta_I(s) \leq K_I(r-\mu_I), 0 \leq s \leq t) \end{aligned} \tag{34}$$

This probability can easily be calculated by simulation, once all parameters of (9) are estimated, that is, $\delta_i(0)$,μ_i, σ_i, as well as the correlations between $\frac{d\delta_i}{\delta_i}$ and $\frac{d\delta_j}{\delta_j}$ (see table 2 for a summary of the methodology). Section 4 illustrates it by an example in the calculation of first-to-default contracts.

3.2 Sensitivity Analysis

We examine the effect of varying the parameters of the dividend process on the formula for debt price (29). We refer the reader to Leland (1994) for similar results obtained when varying the leverage and to section 5 for sensitivity with respect to the exposure to different counterparties. If:

$$\delta_i(0) \quad > \quad \delta_i(t) \tag{35}$$

$$\frac{\delta_i(0)}{(x_i+1)} + \delta_i(0)\ln(K_i(r-\mu_i)) \quad < \quad x_i \tag{36}$$

then, by straightforward differentiation, we see that:

$$\frac{dF_i(t, \mu_i, \sigma_i^2)}{d\mu_i} > 0 \tag{37}$$

$$\frac{dF_i(t, \mu_i, \sigma_i^2)}{d\sigma_i^2} < 0 \tag{38}$$

Reversing the signs of both (35) and (36) reverses the sign of both (37) and (38). For other cases, debt price depends in a more complicated way on parameters.

4 Model Calibration

We see two main steps in the calibration process, in order to calculate debt prices. The first one consists of estimating the fixed network parameters λ. The second one consists of calculating $\hat{\nu}$, the estimator of long-term average operational revenue, and calibrating δ, the payout ratio. Recovery rates $(1 - w_i)$ can be inferred from industry statistics or can be used as fitting parameters.

In some cases, λ is publicly available. Egloff et al (2003) report for instance that "large real estate companies in Switzerland explicitly state their business dependence with the five largest renters in terms of percentage of total income". In other cases, one must use (17), that is, estimate average future expenses from historical data. Referring to (4), two of the components of expenses, namely dividends and debt payments, are easily obtained. The third one, reinvestment C, is harder to calculate well, since it consists of above-average costs. For optimal performance of the model, company-specific accounting data needs to be scrutinized to distinguish between different types of costs[12]. This is clearly a disadvantage of our model if it is meant to be applied mechanically, like the Black-Scholes formula. However, because of the low fitting ability of structural models, we believe that our model should be used preferably by professionals who can apply their specific knowledge of the industry to better distinguish between these costs.

For the second step, it is important to realize that δ should be calibrated under the risk-neutral measure and not the physical measure. In the simple situations where H exhibits little variability, we can immediately calculate the volatility σ_i of the payout from the volatility of dividend and from λ. Similarly, the drift of δ_i in the physical measure can be inferred from the time series of dividends, stock prices, and debt prices: using repeatedly the fact that the ratio S_i/F_i is independent of α_i, and applying Ito's lemma to S_i/F_i, one sees that the ratio of drift of $S_i/\hat{\alpha}_i$ over $S_i/\hat{\alpha}_i$ should equal the ratio of the drift of $F_i/\hat{\alpha}_i$ over $F_i/\hat{\alpha}_i$. We thus look for the value of μ_i that minimizes the following expression:

$$\sum_t \left(\frac{r \left[\frac{\mu_i \delta_i(t)}{r-\mu_i} + \frac{[K_i(r-\mu_i)]^{x_i} \left(\frac{\sum_{j \neq i} \lambda_{ij}}{r} - K_i \right)}{\delta_i^{x_i}(t)} \left(-x_i \mu_i + \frac{x_i(x_i+1)\sigma_i^2}{2} \right) \right]}{\delta_i^{-x_i}(t) \left(x_i \mu_i - \frac{x_i(x_i+1)\sigma_i^2}{2} \right) \frac{w_i \delta_i^{x_i}(0)[K_i(r-\mu_i)]^{x_i}}{\delta_i(0)^{x_i} - [K_i(r-\mu_i)]^{x_i}}} - \frac{S_i(t)}{F_i(t)} \right)^2 \tag{39}$$

We see two different avenues to estimate $\hat{\nu}_i(t)$. The first one consists of taking the average of financial analyst's estimates of operational revenue, and subtract historical fixed costs, hoping that future fixed costs do not need to be risk-neutralized. The second one consists of doing the inference ourselves, that is by inverting the system of equations (27), for $i = 1..I$. This is routinely done in structural models since the flow of information usually goes from stock prices, that are liquidly traded, to debt prices. Table 2 summarizes the particular methodology we followed in the automotive industry example.

For applications to risk-management, as shown in the next section, it is important to know the dynamics of $\hat{\nu}$ in the physical measure. This can be done by historical analysis. A further step consists of regressing the

[12] For the model to have positive flows, the value of fixed production costs should not be superior to the maximum value of observed revenue and (observed total costs + dividends).

time-series of $\hat{\nu}(t)$ against macroeconomic data (such as interest rates, business cycle, GDP, industry-specific statistics) so as to better isolate the "pure" idiosyncratic contagion risk.

4.1 A Specific Example

We analyze a simplified but realistic network of the US automotive industry in which we have taken 8 firms as representative of the overall structure. Figure 2 shows the network, but does not represent (for better visibility) the banking industry, which acts as a sink of debt payments for all 8 firms. Three firms, Autonation, United, and Lithia are car dealers. Ford and GM are car manufacturers. Visteon, Lear, and TRW are part suppliers. The part suppliers extend trade credit to the car manufacturers, who extend trade credit to the car dealers. All borrow from the banking industry. Therefore, contagion will flow from the dealers to the manufacturers to the part suppliers, and will involve the banking industry at all levels. We do not detail the banking industry. It is here interesting to analyze the debt of the manufacturers and part suppliers and look for participation to spreads of the different parties in the network, something other models typically do not reach. For example, we can determine the impact on the spread of a part supplier (Visteon) of a drop in revenues of a car manufacturer (Ford) or of a car dealer (Autonation). As we did not include all firms in the industry, we increased the relevant statistics of each firm proportionally (to their market capitalization), so that the total average statistics, over the segments, equal observed average statistics (see equation (40)). Table 1 shows these annual statistics (before apportioning) from the first quarter 2000 to the fourth quarter 2005. Debt consists of short-term and long-term debt. AP and AR are respectively accounts payable and accounts receivable. Div is annual dividends. We assume that there were no new capital issues during that period for simplification.

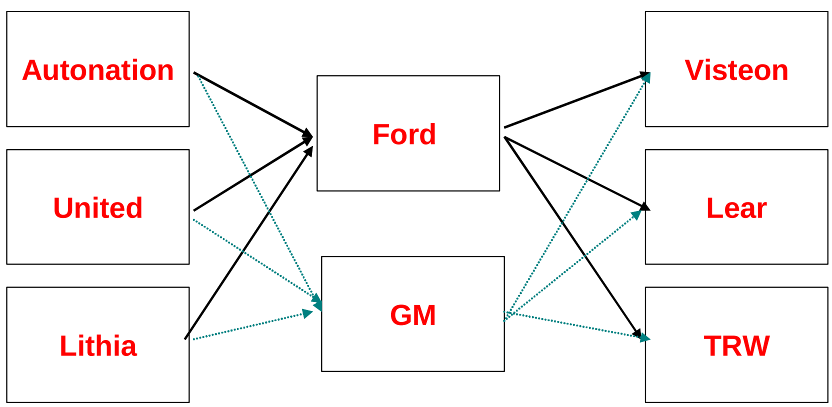

Figure 2: Network in the US automotive industry

Firm	Debt	AR	AP	Revenue	Div	Market cap
Autonation	3092	859	164	19832	0	4126
United Auto	1486	300	147	11188	4	767
Lithia	535	59	1	2356	5	252
Ford	168030	21695	20173	167851	825	29022
GM	186970	76741	26429	186970	722	26880
Visteon	1935	2596	2202	18463	15	1407
Lear	2411	2280	2406	15221	13	2707
TRW	2174	1981	1740	12250	0	2170

Table 1 Average of firm statistics 2000 to 2005 ($ Millions)

To illustrate our apportionment, the principal on the debt issued by Ford to Visteon is:

$$L_{\text{Ford,Visteon}} \approx \text{AR}_{\text{Visteon}} \frac{\text{Market Cap of Ford}}{\text{Market Cap of (Ford+GM)}} \tag{40}$$

As mentioned above, the first step of the calibration requires the most industry-specific knowledge. Since we do not possess it, we decided instead to use the simplest approximation, namely that, on average, above-average production costs are zero, that is:

$$\lim_{t\to\infty} \frac{H_i(t)}{t} \approx \text{average Div+ average (Debt+AP-AR)} * (r + \text{average OAS}) \tag{41}$$

In the right handside of (41), all terms but r refer to firm i. OAS is the option adjusted spread. The average 5-year US Treasury rate over the period 2000 to 2005 was $r = 3.69\%$. Once λ_{ij} was obtained we constructed a time-series of quarterly payout ratios by applying:

$$\frac{\delta_i(t)}{4} = \text{quarterly dividends}_i(t) * \frac{\sum_{j\neq i} \lambda_{ij}}{r} + \sum_{j\neq i} \lambda_{ij} \tag{42}$$

One possible way to assess the power of our model would have been to estimate the parameters over a historical period, say 1995 to 2000, and then compare our theoretical credit spreads with observed spreads over, say 2001 to 2005. Because of the poor performance of structural models at explaining the spread, we took as a benchmark instead a "single-firm" structural model, that is, the model of Goldstein et al (2001). We analyzed Visteon's credit spread. We took as Visteon's recovery rate the value of $(1-w_i)$ that minimized the sum of the squared errors (SSE) between the credit spreads (calculated from the single-firm model) and the observed spreads over the period 2001 to 2005. Our results show that the optimal value if $w_i = 0.61$ for Visteon. Table 2 summarizes the steps we took to estimate the model. We then applied this value to calculate the credit spreads over the same period for our model, which we call hereafter the "multi-firm" model. The observed credit spreads for Visteon, as well as the spreads calculated using both methods are reported in figure 3. Over the period 2001-2005, our model slightly outperformed the benchmark. Indeed, the SSE of the single-firm spreads (compared to the observed spreads) was equal to 132% of the SSE of the multi-firm spreads.

Step	Variable	Calculated by/using
1	λ_{ij}	Relations (40),(41), and (17)
2.1	$\delta_i(0)$	Industry statistics
2.2	w_i	Fitting parameter
2.3	$\delta_i(t)$	Relation (42)
2.4	σ_i	Time series of (42)
2.5	μ_i	Minimizing (39)
2.6	$\hat{\nu}_i(t)$	Analyst estimates

Table 2: The estimation methodology

More interestingly, we can now assess the impact of Ford's revenue variation on Visteon's credit spread as well as the impact of a remote car dealer, in our calculations, Autonation. In our case, a drop in Ford revenue by 50% would impact Visteon's spread by an extra 146 bp (Q3 2005) while an increase in revenue by 50% would impact Visteon's spread by a drop of 191 bp (detailed calculations are available from the authors). A more remote player in a less concentrated segment of the industry, Autonation, affects Visteon less directly. A 50% increase in Autonation revenue will thus impact Visteon's spread by a mere 0.015 bp while a 50% drop would increase the spread by 0.01bp. This type of calculation can easily be done in our model and does not depend on an exogenous correlation or copula but is obtained from the actual economic dependencies between the firms. While being arguably too complex for an immediate application to CDOs, our model provides an interesting alternative to copula models, since it relies on true economic fundamentals rather than on a fit of historical data.

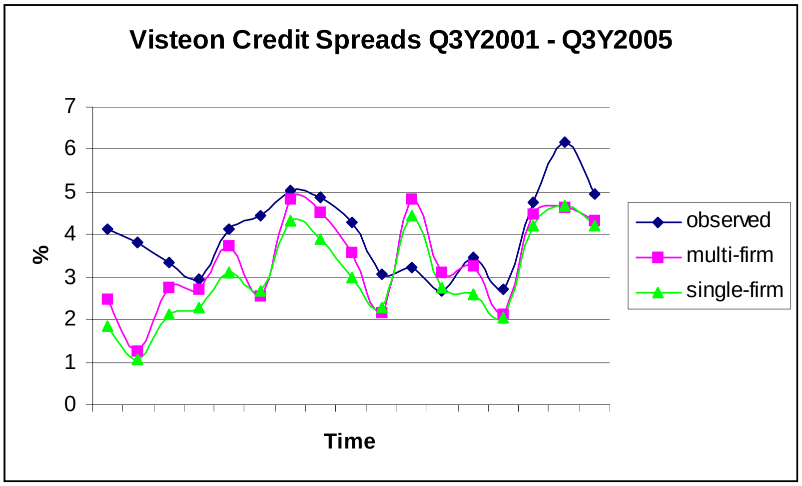

Figure 3: Observed and theoretical credit spreads. The "single firm" model is the benchmark to evaluate the "multi-firm" model.

4.1.1 Valuation of First-to-Default Contracts

Our model allows us to calculate the value $V_{FTD}(t)$ and $V_{STD}(t)$ of the following first-to-default (FTD) and second-to-default (STD) contracts. In this example, the holder of the FTD (STD) contract receives \$100 at time $t+T>0$ if at least one (two) of the firms: Ford, GM, Lithia, and Visteon defaults before time T. We estimated the following parameter values:

i	μ_i (%)	K_i	$\delta_i(0)$
Ford	0.53	16.9	0.921
GM	0.64	18.4	0.903
Lithia	0.25	12.8	0.841
Visteon	-0.02	13.9	0.995

with variance-covariance matrix of log payout rate (i.e., $\frac{1}{dt}Cov^Q[\frac{d\delta}{\delta}, \frac{d\delta}{\delta}]$) given below (in %):

	Ford	GM	Lithia	Visteon
Ford	6.07	3.94	2.02	5.95
GM	3.94	2.97	1.81	4.76
Lithia	2.02	1.81	3.70	3.73
Visteon	5.95	4.76	3.73	8.47

We applied Monte-Carlo simulation (with 10,000 scenarios) to the risk-neutral probabilities of default (34), and obtained:

T	0.5	1	1.5	2	2.5	3
$V_{FTD}(t)$	5.2	5.5	13.2	20.8	27.5	33.4
$V_{STD}(t)$	0.9	1.6	5.1	9.5	13.7	17.8

5 Applications

The major application of our theorem is the linear dependence, in steady state, of equity and debt price on the estimator of long-term revenue rate $\hat{\alpha}_i$. In this section, we apply our theorem to show how market value and cash flow risk are affected by the configuration of the network in some specific examples. We chose to vary the routing matrix λ in such a way that the "leverage" l, defined by:

$$l = \sum_{j \neq i} \lambda_{ij} \quad (43)$$

is the same for all firms. The dynamics of $\hat{\alpha}$ will thus depend only on the number q of counterparties of each firm [13] and on the configuration c of the network.

For expositional simplicity we assume that the estimators of long term operational revenue rate $\hat{\nu}$ are diffusions. Volatilities can be stochastic, but we will be interested only in their value at time $t = 0$. Each firm is exposed, with weight f_i, to one systemic factor S , and with weight $\sqrt{1 - f_i^2}$ to an idiosyncratic factor I_i, where $-1 \leq f_i \leq 1$ and S and I are uncorrelated $\mathcal{F}_t^X$- Brownian motions in the physical measure. So far we have studied network effects only for 10 "similar firms", i.e., 10 firms with identical initial value, drift, and volatility of $\hat{\nu}$.

$$\begin{aligned} \frac{d\hat{\nu}_i(t)}{\hat{\nu}_i(t)} &= \mu^{\hat{\nu}}(\hat{\nu}_i(t), t)dt + \sigma^{\hat{\nu}}(\hat{\nu}_i(t), t)[f_i dS_i + \sqrt{1 - f_i^2} dI_i] \quad (44) \\ \hat{\nu}_i(0) &= \hat{\nu}^0 \end{aligned}$$

From (32) and (44), we see that $\sigma_i^{\hat{\alpha}}$, i.e., the volatility of the estimator of long-term revenue, is a function of $\sigma^{\hat{\nu}}$ and of the routing matrix $\Lambda \equiv (\lambda_{ij})$. Since we want to separate the effects of the configuration of the network and of the number of counterparties, we write it as a function of q and c :

$$\begin{aligned} [\sigma_i^{\hat{\alpha}}(q, c, t &= 0)]^2 dt \equiv Var\left[\frac{d\hat{\alpha}_i}{\hat{\alpha}_i}|_{t=0}\right] \quad (45) \\ &= [\sigma^{\hat{\nu}}(\hat{\nu}^0, 0)]^2 dt \frac{\sum_{j\neq i}[(I - \Lambda)_{ij} f_j]^2 + \left[\sum_{j \neq i}(I - \Lambda)_{ij}\sqrt{1 - f_j^2}\right]^2}{[\sum_{j\neq i}(I - \Lambda)_{ij}]^2} \quad (46) \end{aligned}$$

In order to assess the effect of the number of counterparties, we define the *relative market value* $r_i^X(q, c)$ and the *relative volatility of counterparty risk* $r_i^V(q, c)$ (with respect to having only one counterparty) by:

$$\begin{aligned} r_i^X(q, c) &= \frac{\hat{\alpha}_i(q, c, t = 0)}{\hat{\alpha}_i(1, c, t = 0)} \quad (47) \\ r_i^V(q, c) &= \frac{\sigma_i^{\hat{\alpha}}(q, c, t = 0)}{\sigma_i^{\hat{\alpha}}(1, c, t = 0)} \quad (48) \end{aligned}$$

Observe that in the case of a standard loan with fixed principal, neither r^X nor r^V depend on the type of security (equity or debt). We now analyze two types of network configurations. Note that, by the nature of the chosen examples, r_i^X are the same for all firms, and similarly for r_i^V.

5.1 A Randomized Network

Giesecke and Weber (2004) use the theory of interacting particle systems (IPS) to study credit contagion. When there is an infinity of identical firms on a lattice structure, they prove that the degree of risk decreases with increasing connectivity. We observe the same phenomenon, namely the benefit of diversification.

We simulated c =1..100 scenarios. In each scenario, we randomly varied the configuration of the network, i.e., the identity of the counterparties for fixed number of counterparties (for each firm) q. We focussed on analyzing only one specific firm. We observed the following results. First, the relative market value of

[13] For simplicity in our examples we assume that each firm borrows the same proportion from each of its lenders, that is, λ_{ij} is equal to either $\frac{l}{q}$ or zero.

debt does not vary with q. As a consequence, a risk-averse investor always prefers a security with a lower volatility, supposing no systemic risk (i.e., $f_i = 0$ in (44)), and costless portfolio rebalancing. Second, the average relative volatility of debt, defined below, is always decreasing with the number of counterparties:

$$\text{avg rel vol of debt}(q) = \frac{1}{100} \sum_{c=1}^{100} r^V(q,c)$$

Third, as expected, the maximum benefit of diversification (with respect to instantaneous risk) is obtained when a firm and its counterparties have opposite exposure to the systemic factor, i.e., $f_i = -f_j$ for all $j \neq i$. Figure 4 shows our results in the case without systemic risk for different values of "leverage" l.

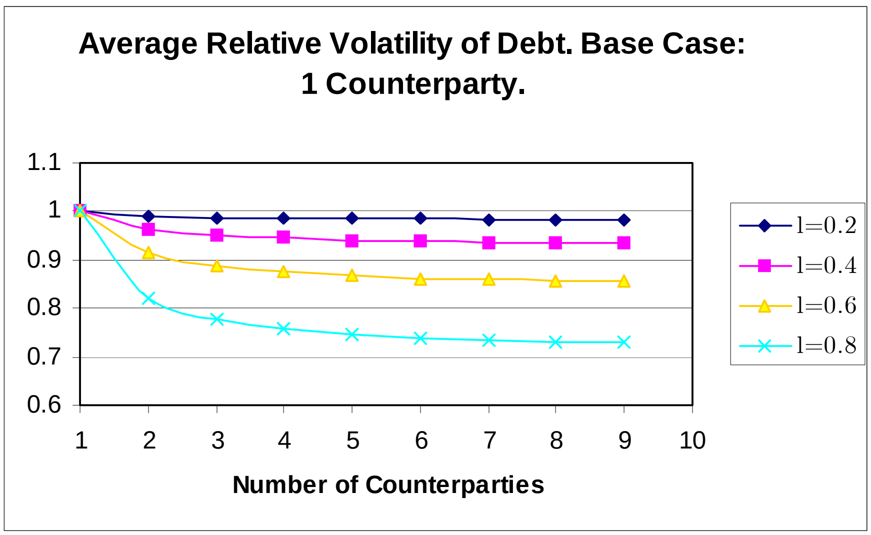

Figure 4: Randomized network without systemic risk for different "leverage" l.

5.2 A Cyclic Network

In this section we analyze the same model as in the previous section, but fix one particular network structure, namely a cycle of 10 firms. In such a network, each firm borrows from the q closest firms along the cycle. The results are strikingly different from the randomized network case, as figure 5 shows, in the case without systemic risk ($f_i = 0$). The relative volatility of debt has a minimum at around 3 or 4 counterparties. This optimum seems fairly robust when we vary the exposure to systemic risk. We observe regularly decreasing curves in less than 50% of all possible combinations of exposures, all others showing a minimum at 3 or 4 counterparties. As in the previous subsection, the relative market value of debt does not vary with the number of counterparties.

For equilibrium to be stable, it is logical to assume that this peculiar effect should be "arbitraged" away. We propose two possible explanations for this apparent arbitrage opportunity: (i) the cost of maintaining relationships (which we have not modelled so far) may not be a monotonous function of the number of counterparties, (ii) the principal loaned to firms maintaining an optimum number of counterparties should be smaller than the principal loaned to other firms, for the same amount of debt payment.

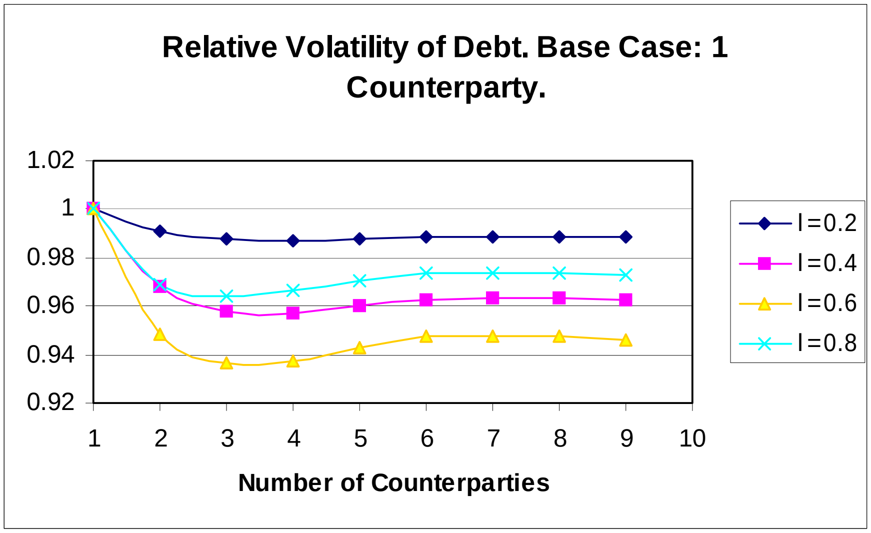

Figure 5: Cyclic network without systemic risk for different "leverage" l.

6 Conclusions

We developed a structural model of default to analyze credit risk under general lending/borrowing relationships in a network economy. We derived the value of the firm and of equity when default is affected by the default of other parties in the network, even those parties that have no direct economic link to the firm considered. We characterized a combination of primitives of the economy which makes it particularly robust to one aspect of contagion, i.e., the game-theoretic behavior of strategic equityholders. We implemented the model on a simplified version of the US automotive industry. We applied our theory to show that, in general, firms should seek maximum diversification of counterparties. However, we found a specific structure of counterparty network, namely a cyclic network, where there is a finite optimum number of counterparties, with respect to instantaneous risk. This could lead researchers to investigate what type of network optimizes counterparty exposure.

Our model has some clear limitations. Our economic model is a stylized model of firm relationships. Our pricing formula is valid only in steady-state, and it is difficult to assess the speed of convergence to steady state. Nevertheless, the validity of these assumptions is quantifiable, although not in closed-form. For future research, we intend to analyze more specifically the effects of network structure on volatility and credit dependencies.

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Appendix

Steady State

DEFINITION *Steady state is reached when for $\bar{\nu}$ fixed*
(i) the increments of R and of $C + D$ are stationary in the Q measure and
(ii) Z is stationary in the N measure, given a realization of $\bar{\nu}$.

Note that proposition 3 uses only condition (i), while proposition 1 uses only condition (ii), so that both conditions are needed in the theorem. Since our results hold in steady state, we need to address [14] the issue of speed of convergence to steady state.

Condition (ii) is equivalent to stationarity of a Jackson network of M/M/∞ queues. In some particular cases, exponential ergodicity can be proved. In other terms, the total variation norm of the difference between the $n-$step transition probabilities of the embedded Markov chain and the stationary distribution is bounded by, say, $c\alpha^n$ (given an initial state). Hordijk and Spieksma (1992) show that it is the case for the two-centre open Jackson network. For a type a single queue, Spieksma (1992) mentions that α is the second largest eigenvalue of the embedded Markov chain, and provides explicit values for c, which are in full generality difficult to calculate. Manita (1996) describes the convergence time to equilibrium of a Jackson network of $M/M/1/N$ finite buffer capacity queues. A more practical calculation stems from the diffusion approximation (see e.g., Reed and Ward (2004)). In this case, the convergence rate is also exponential, and is dictated by the largest eigenvalue of the P matrix of proposition 2.

As for condition (i), theorem 1 in Korolev (1998) shows that (normalized) Cox processes converge weakly to a random variable only if the (normalized) controlling process also converges weakly to another random variable, that is, in our notation, only if:

$$\lim_{T\to\infty}\int_0^T \frac{k_i(t)}{d_i(t)}dt \Longrightarrow U_i$$

where U_i is a random variable and d_i a function of time. It is well-known that $k_i(t)$ has a chi-square distribution, and the speed of convergence can be calculated numerically. The formula for the first moment is indeed:

$$E^Q[k_i(t)|\bar{\nu}_i] = 1 + (k_i(0) - 1)\exp(-\frac{t}{m_i})$$

showing that the convergence rate is inversely proportional to $\frac{1}{m_i}$.

Assumption (A9)

This type of assumption is growing in popularity in the finance literature (see e.g., Jarrow and Madan (1999)). Assumption (A9) states that when a flow is almost constant (in our case, debt payment), its precise value has almost no impact on the optimal default policy. Let $\mathcal{T}(X)$ be the set of $\sigma(X)$-measurable stopping times. The value V_t of a claim that pays cash flows per unit of time $f(X(s))$ up to default time $T \in \mathcal{T}(X)$ can then be written, in accordance with (1):

$$V_t(\mathcal{T}(X)) = \max_{T\in\mathcal{T}(X)} E^Q[\int_t^T \exp(-r(s-t))f(X(s))ds|\mathcal{F}_t^X] \tag{49}$$

We define another set of state variables (^{t} denotes transpose):

$$Y^t(t) = \left[\begin{array}{ccccc} C^t(t) & D^t(t) & \delta^t(t) & E^Q[P^t(t)|\bar{\nu}] & M^t(t) \end{array}\right] \tag{50}$$

[14] We thank an anonymous referee for pointing out this fact.

(A9) *(Regularity of the optimal default policy). Suppose that, for $0 \leq t \leq T^f$.*

$$Var^Q[C_i(t) + D_i(t)|\bar{\nu}] >> Var^Q[P_i(t)|\bar{\nu}] = O(\frac{1}{\sqrt{n}})$$

Then,

$$V_t(\mathcal{T}(X)) - V_t(\mathcal{T}(Y)) = O(\frac{1}{\sqrt{n}}) \tag{51}$$

Results from Queueing Theory

The following proposition is a loose[15] combination of well-known results from queueing theory (see Glynn and Whitt (1991) for the speed of convergence, and Mandelbaum and Pats (1998) for the main result).

PROPOSITION 2. *Let*

$$P = \begin{bmatrix} p_1 & -\lambda_{1,2}p_1 & . & -\lambda_{1,n}p_1 \\ -\lambda_{2,1}p_2 & . & . & . \\ . & . & . & \lambda_{n-1,n}p_{n-1} \\ -\lambda_{n,1}p_n & .. & \lambda_{n,n-1}p_n & p_n \end{bmatrix}$$

Conditional on $Z^{(n)} > 0$, the distribution of $Z^{(n)}$ in a Jackson network of $M/M/\infty$ queues satisfies

$$Q(\frac{Z^{(n)}(t) - \bar{Z}}{\sqrt{n}} \leq z) = Q(Z^c(t) \leq z) + o(\frac{1}{\sqrt{n}}) \tag{52}$$

where Z^c follows:

$$dZ^c(t) = -PZ^c(t)dt + \Sigma dW^Q \tag{53}$$

*for some matrix Σ which is a function of only $p, \nu,$and λ, W^Q I-dimensional Brownian motion, and $\bar{Z}$ given by (*24).

When $I = 1$, the matrix Σ^2 is equal to 2ν.

PROPOSITION 3. *Consider a network of Cox/M/∞ queues with Bernoulli routing, as described in section 2.2. Let b be any vector with finite components larger than one. Suppose $p_i \leq \frac{1}{8b_iT^f}$. In steady-state, there exists a vector m so that, for all n and all $t \leq T^f$:*

$$b_i Var^Q[\frac{P_{ij}^{(n)}}{\lambda_{ij}}(t)|\bar{\nu}] \quad \leq \quad Var^Q[R_i^{(n)}(t)|\bar{\nu}] \tag{54}$$

$$Var^Q[\frac{P_{ij}^{(n)}(t)}{n}|\bar{\nu}] \quad = \quad O(\frac{1}{n}) \tag{55}$$

The bound on p_i is quite conservative; it can be increased, e.g., if we restrict proposition 3 to hold only for all $n \geq N$ for some fixed N. Note that, for $P_{ij}^{(n)}$ to exhibit less variability than $R_i^{(n)}$, one needs a small value of p_i. However, a small value of p_i implies by (24) a large average value of the cash account.

PROOF: Let $V_{k,i}^{(n)}$ be the sequence of interarrival times of $R_i^{(n)}$. We apply [16] the central limit theorem for stationary sequences (theorem 7.7.6 Durrett (1991)):

$$Var^Q[\frac{R_i^{(n)}(t)}{\sqrt{n}}|\bar{\nu}_i] \to c_i^2(m_i)t \tag{56}$$

[15] The error term is indeed $o(n^{-(p-1)/2(p+1)})$ for any finite p.
[16] Strictly speaking, the theorem applies to the process $\tilde{R}$ obtained from R by linear interpolation.

and:

$$c_i^2(m_i) = Var^Q[V_{0,i}^{(1)}|\bar{\nu}_i] + 2\sum_{k=1}^{\infty} Cov^Q[V_{0,i}^{(1)}, V_{k,i}^{(1)}|\bar{\nu}_i] \tag{57}$$

The following lemma is proved in the companion paper.

Lemma 1 $Cov^Q[V_{0,i}^{(1)}, \sum_{k=1}^{\infty} V_{k,i}^{(1)}|\bar{\nu}_i] \geq 0$

Therefore

$$c_i^2(m_i) \geq Var^Q[V_{0,i}^{(1)}|\bar{\nu}] = -\int_0^{\infty} t^2 dP_i(t) - (\int_0^{\infty} t dP_i(t))^2 \tag{58}$$

$$P_i(t) = E^Q[\exp \int_0^t (-\nu_i(s)ds)|\bar{\nu}_i] \tag{59}$$

Lemma 2, which is also proved in the companion paper, is a consequence of the formula for $P_i(t)$ (see, e.g. Cox et al (1985)).

Lemma 2: *Provided* m_i *is larger than the maximum value of* $\{\sqrt{\frac{8}{\sqrt{2}}}, (\frac{4\bar{\nu}_i \ln 4}{\ln \frac{9}{8}})^{1/3}, \frac{2\bar{\nu}_i}{\ln \frac{4}{3}}, \frac{\bar{\nu}_i + \sqrt{\bar{\nu}_i^2 + 2\bar{\nu}_i \ln 2}}{\ln 2}, \sqrt{4\bar{\nu}_i}\}$, *then:*

$$Var^Q[V_{0,i}^{(1)}|\bar{\nu}_i] \geq \frac{m_i^2}{4\bar{\nu}_i} \tag{60}$$

Clearly, $c_i^2(0) = \bar{\nu}_i$, so by continuity of c_i^2 , (60) shows that $c_i^2(m_i)$ can take all possible values larger than $\bar{\nu}_i$. Theorem 4 in Whitt (1984) gives a formula for the steady-state variance of departures in any infinite-server queue with general arrivals with mean n and mean service rate 1. By investigating the original result in Borovkov (1967) it is clear that Whitt's result holds for mean service rate $p_i \neq 1$, but the mean arrival rate $\bar{\nu}$ has to be incorporated followingly: the variance of the term $Y_1(t)$ in Whitt has to be multiplied by ν. To prove the proposition we would need results for the variance of arrivals (56) and departures for finite n, but unfortunately the rate of convergence is not known. Therefore we introduce a very conservative rate of convergence: n^{β}, where β is an unknown positive number strictly smaller than one. Theorem 4 in Whitt (1984) then spells:

$$Var^Q[\frac{P_{ij}^{(n)}(t)}{\lambda_{ij}\sqrt{n}}|\bar{\nu}] \leq \bar{\nu}_i t + [c_i^2(m_i) - \bar{\nu}_i]\frac{[e^{pt} - 1]^2}{2p} + Kn^{\beta}t \tag{61}$$

for some $K > 0$. Likewise we rewrite (56):

$$Var^Q[\frac{R_i^{(n)}(t)}{\sqrt{n}}|\bar{\nu}_i] \geq c_i^2(m_i)t - Kn^{\beta}t \tag{62}$$

Since $p_i \leq \frac{1}{8b_i T^f}$:

$$Var^Q[\frac{P_{ij}^{(n)}(t)}{\lambda_{ij}\sqrt{n}}|\bar{\nu}] \leq \bar{\nu}_i t + 4[c_i^2(m_i) - \bar{\nu}_i]p_i t + Kn^{\beta}t \tag{63}$$

And the proposition holds because $c_i^2(m_i)$ can take any arbitrary large value.

Proof of Theorem

Lemma 3: *For $T^f \geq T \geq t$:*

$$E^Q[C_i(T)+D_i(T)|\mathcal{F}_t,\bar{\nu}] = E^N[C_i(T)+D_i(T)|\mathcal{F}_t,\bar{\nu}] + O(\frac{1}{\sqrt{n}}) \tag{64}$$

To calculate equity price, we neglect dividends incoming after the final time T^f, which is assumed very large. By (1), and (20), $S_i(t)$ is then equal to (for $t \leq T_i^1$ the first bankruptcy time):

$$\max_{T_i \in \mathcal{T}(X)} E^Q[E^Q[\int_t^{T_i} e^{-r(s-t)} \frac{\delta_i(s) - \sum_{j\neq i}\lambda_{ij}}{1-\sum_{j\neq i}\lambda_{ij}} d(C_i(s)+D_i(s))|\mathcal{F}_t^X,\bar{\nu}]|\mathcal{F}_t^X] =$$

$$\max_{T_i \in \mathcal{T}(Y)} E^Q[E^Q[\int_t^{T_i} e^{-r(s-t)} \frac{\delta_i(s) - \sum_{j\neq i}\lambda_{ij}}{1-\sum_{j\neq i}\lambda_{ij}} d(C_i(s)+D_i(s))|\mathcal{F}_t^Y,\bar{\nu}]|\mathcal{F}_t^X] + O(\frac{1}{\sqrt{n}}) =$$

$$\max_{T_i \in \mathcal{T}(Y)} E^Q[E^N[\int_t^{T_i} e^{-r(s-t)} \frac{\delta_i(s) - \sum_{j\neq i}\lambda_{ij}}{1-\sum_{j\neq i}\lambda_{ij}} d(C_i(s)+D_i(s))|\mathcal{F}_t^Y,\bar{\nu}]|\mathcal{F}_t^X] + O(\frac{1}{\sqrt{n}}) =$$

$$\max_{T_i \in \mathcal{T}(Y)} E^Q[E^N[\int_t^{T_i} e^{-r(s-t)} \frac{\delta_i(s) - \sum_{j\neq i}\lambda_{ij}}{1-\sum_{j\neq i}\lambda_{ij}} d(C_i(s)+D_i(s))|\delta_i(t),\bar{\nu}]|\mathcal{F}_t^X] + O(\frac{1}{\sqrt{n}}) =$$

$$E^Q[\bar{\alpha}_i|\mathcal{F}_t^X] \max_{T_i \in \mathcal{T}(Y)} E^Q[\int_t^{T_i} e^{-r(s-t)}[\delta_i(s) - \sum_{j\neq i}\lambda_{ij}]ds|\delta_i(t)] + O(\frac{1}{\sqrt{n}}) =$$

$$\hat{\alpha}_i(t) \max_{T_i \in \mathcal{T}(\delta_i)} E^Q[\int_t^{T_i} e^{-r(s-t)}[\delta_i(s) - \sum_{j\neq i}\lambda_{ij}]ds|\delta_i(t)] + O(\frac{1}{\sqrt{n}})$$

The first equality follows by (5) and assumption (A9), the second one follows by lemma 3, the third one by proposition 1 and the fact that δ_j is not causally related to $C_i + D_i$, and the fourth one by independence of $\bar{\nu}$ and δ. The optimal stopping problem of firm i is now approximately decoupled from the problem of the other firms. Following Goldstein et al (2001), the optimal bankruptcy time T_i^1 is the first passage time of $\delta_i(t)$ under a fixed barrier, and we use their formulas to obtain (27). Due to assumption (A6) on restructuring costs, δ_i takes the same value at each default time, so that the interdefault times are identically distributed. As a result, the value of the cumulated bankruptcy costs becomes:

$$\begin{aligned} TBC_i(t) &= E^Q[\sum_{k\geq 1} e^{-r(T_i^k - t)} BC_i^k|\mathcal{F}_t^X] \\ &= w_i E^Q[\sum_{k\geq 1} e^{-r(T_i^k - t)} \int_{T_i^k}^{\infty} e^{-rs} dA_i(s)|\mathcal{F}_t^X] \\ &= \hat{\alpha}_i(t) \frac{w_i E^Q[\exp(-rT_i^1)|\delta_i(t)]}{r(1-E^Q[\exp(-rT_i^1)|\delta_i(0)])} + O(\frac{1}{\sqrt{n}}) \end{aligned}$$

and (28) follows from (5.5) in Karlin and Taylor (1975) p. 362.

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