# ESSENTIAL MATH REVIEW FOR ENGINEERING

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*Dedicated to Professor David A Taggart for my first class in electrical engineering and Professor Xinwei Yu for all the help throughout my math courses.*

Abstract. This review is created as a quick math refresher for college math /science /engineering students especially Junior level or above. It is short, precise, and neat presentation of many essential math concepts that are applied to engineering. Not only does the review examine the very basic of math which is system of numbers and trigonometry but it also includes a comprehensive presentation of calculus. Beyond calculus, we will discuss Fourier Series in great details. In summary, this review may be simply referred as engineering math in a nutshell.

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## 1. **Introduction**

Having recently graduated from UCLA as a Math–Applied Science major in June 2008, I really enjoy applying abstract math to solve practical problems. Throughout 4 years in college, I have taken many courses in various disciplines including, but not limited to, art, social science, science, statistics, and business. Specifically, I have finished about 10 rigorous math courses from basic calculus to upper division ordinary differential equations. In addition to that, I have also completed 5 accounting, 3 management science (2 of which are MBA graduate level), 2 applied statistics and 1 real estate finance courses. To satisfy my desire to apply math to other fields even more, I registered for this electrical engineering course in this summer through UCLA Extension. Furthermore, I am working toward obtaining MS in Engineering Management from USC and MS in Engineering in Computer Network from UCLA.

Having this opportunity to submit a term project for this summer course, I decide to make a comprehensive math review of many important concepts that are frequently used in engineering. First, I will go through briefly the system of number as the foundation of math, then review trignometry especially the identities. With these 2 tools, we can start reviewing more advanced topics in calculus. Beyond calculus, we will also go to examine Fourier Series.

I try to make this review as comprehensive as possible, as short as possible but yet as clear as possible as maximizing essential details and minimizing redundancy. The main audience are presummably undergraduate college juniors or seniors who major in technical fields like math, science or engineering and who have finished at least some basic calculus that cover infinite series. I strongly believe that graduate students especially engineering graduates will also find this review helpful and they can refer to this to quickly and efficiently refresh their math knowledge.

In this review, I sometimes go through a long proof like the case in constructing the coefficients of the Fourier Series and I sometimes just list the properties of the concepts without any proof. Finally, I really hope that you enjoy reading this review as much as I enjoy writting it and I will appreciate any comment regarding this review. You can directly contact me at kiettran2006@ucla.edu.

## 2. **System of Numbers**

(1) **Natural numbers** consist of set of whole numbers including 0 such as {0,1,2,3,...}and denoted by N.

(2) **Integers** consist of a set of both positive and negative whole numbers including 0 such as{...,-1,-2,-3,0,1,2,3,...} and denoted by Z.

(3) **Rational numbers** consist all numbers that can be expressed as a fraction such a $\frac{m}{n}$ where m and n are integers. Beside, integers is a subset of rational numbers. Rational numbers set is denoted by Q.

(4) **Real numbers** consist of the set of rational numbers plus the irrational numbers which can not be expressed in the fraction $\frac{m}{n}$ where m and n are integers.

**Example 1.** *All transcendental numbers are irrational numbers such as* $\Pi^r$, $e^r$, $e^{\Pi}$ *etc where* $r \neq 0$ *and* $r$ *is rational number.*

Irrational numbers have some special properties: If we express them in a demimal expansion format, the expansion will never repeat or terminate while the decimal expansion of rational numbers have recurring decimal.

**Example 2.** *For instance, the rational number* $0.\overline{9} = 0.999...$ *where the 9 repeats indefinitely is just another expression of 1.*

*Proof.* let $0.999... = x$, then $10x = 9.999...$ and $10x - 1x = 9x = 9.999... - 0.999... = 9$ so $x = 1$ ☐

*However, the irrational number* $\pi = 3.14159265358979323846...$ *does not have any recurring decimal pattern and it never terminates.*

**Example 3.** *To illustrate a difference between rational and irrational number, let us refer to one example in which we calculate the length of the hypothenuse of a right triangle (let it be x) with the 2 sides having length of 1 meter. By the Pythagorean theorem, we will have* $x^2 = 1^2 + 1^2 = 2$ *so* $x = \sqrt{2}$. *Why can't we express* $x$ *or* $\sqrt{2}$ *as a fraction?*

*Proof.* Assume that we can express $\sqrt{2}$ as a fraction $\frac{m}{n}$ where $m$ and $n$ are integers and this fraction is in lowest term, then $\sqrt{2}^2 = (\frac{m}{n})^2$. Therefore, $m^2 = 2n^2 \iff m$ is even. Let $m = 2k$ where $k$ is some integer, then $(2k)^2 = 2n^2 \iff n^2 = 2k^2 \iff n$ is even. If both $m$ and $n$ are even, then $\frac{m}{n}$ is not in lowest term. We encounter a contradiction and therefore $\sqrt{2}$ can not be expressed in fraction and $\sqrt{2}$ is an irrational number. ☐

**Summary 1.** *Both rational and irrational numbers make real number set a complete set and denoted by R. In simple language, if we draw a straight and continuous line, theoretically speaking, we can measure any exact distance between two points and that distance can be expressed as a real number (either rational or irrational). Practically speaking in reality, we don't have any equipment that can give us the exact measurement nor can we express the exact measurement in expanding decimal format because many (not all) real numbers have infinite decimal expansion.*

(5) **Complex numbers** consist of real numbers and imaginary numbers or the combination of the two. Its representation is usually in the rectangular (cartesian) form of $z = a + bi$ where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$. We can denote the real part as $a$=Re(z) (horizontal axis) and and the imaginary part as $b$=Im(z) (vertical axis). It also has polar form of $z = re^{i\theta}$; $r > 0$ and $r \in \mathbb{R}$ is referred as the magnitude of $z$ and $\theta$ is referred as the phase of $z$.

With Euler's formula: $e^{i\theta} = cos\theta + isin\theta$, then the relationships between rectangular and polar representation of $z$ are:
$a = r\cos\theta, b = r\sin\theta, r = \sqrt{a^2 + b^2}, \theta = \tan^{-1}\frac{b}{a}$

Therefore, $z^n = r^n e^{in\theta} = r^n(\cos n\theta + i\sin n\theta)$ and we have DeMoivre's relation $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

## 3. Trigonometry

### 3.1. Basic and interesting tricks.

**Example 4.** *1 radian* $= \frac{180}{\pi}$ *degrees, 1 degree* $= \frac{\pi}{180}$ *radian*

**Example 5.** *We can call the quadrant I, II, III and IV as: A-S-T-C (All-Student-Take-Calculus) in which it implies All (A) sine, cosine, tangent are positive in quadrant I (range from 0-90 degree), only Sine (S) is positive in quadrant II (90-180 degree), only Tangent (T) is positive in quadrant III (180-270 degree), and only Cosine (C) is positive in quadrant IV (270-360 degree).*

### 3.2. Trigonometry identities.

#### 3.2.1. *Pythagorean Identities.*

$$\sin^2\theta + \cos^2\theta = 1 \tag{3.1}$$

$$\tan^2\theta + 1 = \sec^2\theta \tag{3.2}$$

$$1 + \cot^2\theta = \csc^2\theta \tag{3.3}$$

#### 3.2.2. *Double Angle Formulas.*

$$\sin(2\theta) = 2\sin\theta\cos\theta \tag{3.4}$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta \tag{3.5}$$

$$\tan(2\theta) = \frac{2\tan\theta}{1-\tan^2\theta} \tag{3.6}$$

#### 3.2.3. *Half Angle Formulas.*

$$\sin^2\theta = \frac{1}{2}(1-\cos(2\theta)) \tag{3.7}$$

$$\cos^2\theta = \frac{1}{2}(1+\cos(2\theta)) \tag{3.8}$$

$$\tan^2\theta = \frac{1-\cos(2\theta)}{1+\cos(2\theta)} \tag{3.9}$$

#### 3.2.4. *Sum and Difference Formulas.*

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \tag{3.10}$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \tag{3.11}$$

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta} \tag{3.12}$$

### 3.2.5. *Product to Sum Formulas.*

$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)] \tag{3.13}$$

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)] \tag{3.14}$$

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)] \tag{3.15}$$

$$\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)] \tag{3.16}$$

### 3.2.6. *Sum to Product Formulas.*

$$\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \tag{3.17}$$

$$\sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) \tag{3.18}$$

$$\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) \tag{3.19}$$

$$\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right) \tag{3.20}$$

## 3.3. Trigonometry Functions Properties.

### 3.3.1. *Even Functions.*

**Definition 1.** *Some function $f(x)$ is called an **even function** when $f(-x) = f(x)$. Furthermore, if $f(x)$ is differentiable, its derivative is an odd function.*

**Example 6.** $\cos(-\theta) = \cos\theta$

**Example 7.** $\sec(-\theta) = \sec\theta$

**Example 8.** *$ax^n$ where $n$ is an even integer and $a$ is a real constant.*

**Example 9.** *A combination of even functions is also an even function. For example, $\sum_{i=0}^{i=N} C_i X^i$ where index $i$ is a sequence of even natural number only and $C_i$ is some constant for $X^i$.*

### 3.3.2. *Odd Functions.*

**Definition 2.** *Some function $f(x)$ is called an **odd function** when $f(-x) = -f(x)$. Furthermore, if $f(x)$ is differentiable, its derivative is an even function.*

**Example 10.** $\sin(-\theta) = -\sin\theta$

**Example 11.** $\tan(-\theta) = -\tan\theta$

**Example 12.** $\csc(-\theta) = -\csc\theta$

**Example 13.** $\cot(-\theta) = -\cot\theta$

**Example 14.** *$bx^n$ where $n$ is some odd integer and $b$ is a real constant.*

**Example 15.** *A combination of odd functions is also an odd function. For example,* $\sum_{i=0}^{i=N} C_i X^i$ *where index* $i$ *is a sequence of odd natural number only and* $C_i$ *is some constant for* $X^i$.

### 3.3.3. ***Periodic Functions.***

**Definition 3.** *A **periodic function** is a function that repeats its values after some definite* period *has been added to its independent variable. Mathematically,* $f(x) = f(x + np)$ *for* $n \in \mathbb{N}$ *and* $p$ *is the* period.

**Example 16.** *All trigonometry functions are periodic. Sine, Cosine, CoSecant, and Secant functions are periodic with period of* $2\pi$ *while Tangent and CoTangent functions are periodic with period of* $\pi$.

## 4. Calculus

### 4.1. Limit.

**Definition 4.** *Limit of some function* $f(x)$ *is* $L$ *means as* $x$ *approaches* $a$ $\lim_{x \to a} f(x) = L$ *provided we can make* $f(x)$ *as close to* $L$ *as we want for all* $x$ *sufficiently close to* $a$*, from both sides, without actually letting* $x$ *be* $a$.

### 4.2. Continuity.

**Definition 5.** *A function* $f(x)$ *is **continuous** at x=a if* $\lim_{x \to a} f(x) = f(a)$. *In addition,a function is continuous on the interval [a, b] if it is continuous at every point in the interval.*

### 4.3. Differentiation.

**Definition 6.** *A derivative is a measurement of how a function changes when the values of its inputs change or simply the rate of change of a function. In addition, differentiation is a method to compute the rate at which a dependent output y, changes with respect to the change in the independent input x.*
*Mathematically, the derivative of function* $f$ *at* $a$ *is defined as*

$$f^{'}(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

### 4.3.1. ***Common Derivatives.***

| $\mathbf{f(x)}$ | $\mathbf{f'(x)}$ | remarks |
|---|---|---|
| $x^n$ | $nx^{n-1}$ | |
| $\exp^x$ | $\exp^x$ | |
| $a^x$ | $\ln(a)a^x$ | |
| $\ln(x)$ | $\frac{1}{x}$ | $x > 0$ |
| $\log_a(x)$ | $\frac{1}{x \ln(a)}$ | |
| $\sin(x)$ | $\cos(x)$ | |
| $\cos(x)$ | $-\sin(x)$ | |
| $\tan(x)$ | $\sec^2(x)$ | |
| $\sec(x)$ | $\sec(x)\tan(x)$ | |
| $\csc(x)$ | $-\csc(x)\cot(x)$ | |
| $\cot(x)$ | $-\csc^2(x)$ | |
| $\arcsin(x)$ | $\frac{1}{\sqrt{1-x^2}}$ | |
| $\arccos(x)$ | $-\frac{1}{\sqrt{1-x^2}}$ | |
| $\arctan(x)$ | $\frac{1}{\sqrt{1+x^2}}$ | |

4.3.2. *Differentiation Rules.* Let $f(x)$, $g(x)$ and $h(x)$ be some functions with single variable $x$ and $a, b$ be some constant, then

- If $f(x)$ =constant $\Longrightarrow f' = 0$. **(Constant Rule)**
- $(af + bg)' = af' + bg'$. **(Sum Rule)**
- $(fg)' = f'g + fg'$. **(Product Rule)**
- $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$. **(Quotient Rule)**
- If $f(x) = h(g(x)) \Longrightarrow f'(x) = h'(g(x))(g'(x))$. **(Chain Rule)**

### 4.4. Integration.

**Definition 7.** *Integration is the process of finding the integral. Formally, given a function $f(x)$ of a real variable $x$ and an interval $[a, b]$ of the real line, the integral $\int_a^b f(x)dx$ is equal to the area of a region in the xy-plane bounded by the graph of $f(x)$, the $x$-axis, and the vertical lines $x = a$ and $x = b$, with areas below the $x$-axis being subtracted.*

Loosely speaking, integration can be thought as the reverse of differentiation. For example, you start out with some function $f(x)$, then you differentiate that function and get $f'(x)$, and finally you integrate the result $f'(x)$ with respect to $x$, you end up with the original function $f(x)$. Hence, integral is also called **anti-derivative**.

4.4.1. ***Common Integrals.***

| $\mathbf{f(x)}$ | $\int \mathbf{f(x)dx}$ | remarks |
|---|---|---|
| $k$ | $kx + c$ | $k, c$ are constants |
| $x^n$ | $\frac{x^{n+1}}{n+1} + c$ | except $n \neq -1$ |
| $\frac{1}{x}$ | $\ln\lvert x\rvert + c$ | |
| $\exp^x$ | $\exp^x + c$ | |
| $a^x$ | $\frac{a^x}{\ln a} + c$ | |
| $\ln(x)$ | $x\ln(x) - x + c$ | |
| $\sin(x)$ | $-\cos(x) + c$ | |
| $\cos(x)$ | $\sin(x) + c$ | |
| $\tan(x)$ | $\ln\lvert\sec(x)\rvert + c$ | |
| $\sec(x)\tan(x)$ | $\sec(x) + c$ | |
| $\csc(x)\cot(x)$ | $-\csc(x) + c$ | |
| $\csc^2(x)$ | $-\cot(x) + c$ | |

4.4.2. ***Integration Techniques.*** Integration is more difficult and trickier than differentiation. That is why mathememeticians develop some methods to integrate certain functions but still there are many functions that can't simply be integrated using these known methods. In those difficult cases, we rely on numerical analysis in which we approximate the results since we can not have an exact solution. We will now briefly go through some well known integration techniques below.

**Theorem 1. *Integration By Parts***
*Let $u = f(x)$ and $v = g(x)$*
*then $du = f'(x)dx$ and $dv = g'(x)dx$*

$$\int udv = uv - \int vdu$$

**Example 17.** *Evaluate* $\int xe^{2x}dx$
*Let* $u = x$ *and* $v = \int e^{2x}dx = \frac{e^{2x}}{2}$
*then* $du = dx$ *and* $dv = e^{2x}dx$

$$\int xe^{2x}dx$$

$= \frac{xe^{2x}}{2} - \int \frac{e^{2x}}{2}dx$ *and finally,*

$$\int xe^{2x}dx = \frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + C$$

*with constant* $C$.

**Theorem 2.** ***Integration by U-Substitution***
*Let* $I \in R$ *be a real interval and* $g : [a, b] \to I$ *be a continuously differentiable function. Suppose that* $f : I \to R$ *is a continuous function. Then*

$$\int_a^b f(g(t))g'(t)dt = \int_{g(a)}^{g(b)} f(x)dx$$

Note that $x = g(t)$ yields $\frac{dx}{dt} = g'(t)$ and hence $dx = g'(t)dt.$

This method might involve trial and error because for some functions, sometimes no matter what substitution we make, we can't derive a solution.

However, this method is relatively easier than other methods. With consistent practise, we can be more efficient with this method as we know immediately which substitution we should make and learning this is best through examples.

In general, this method works best for trigonometry functions and the functions involving roots. Let's go through some typical examples and some of these may be difficult and tricky to get started because it requires some tricks before we can apply the substitution.

**Example 18.** *Evaluate* $\int \tan(x)dx$

$$\int \tan(x)dx$$

$= \int \frac{\sin(x)}{\cos(x)}dx$ *and let* $u = \cos(x)$ *so* $du = -\sin(x)$ *so*

$$\int \tan(x)dx = -\int \frac{1}{u}du$$

$= -\ln|u| + c$
$= -\ln|\cos(x)| + c$ *since* $u = \cos(x)$
$= \ln|cos(x)|^{-1} + c$ *and finally,*

$$\int \tan(x)dx = \ln|\sec(x)| + c$$

*where* $c$ *is some constant.*

**Example 19.** *Evaluate* $\int \tan^3(x)dx$

$$\int \tan^3(x)dx$$

$= \int \tan(x)\tan^2(x)dx$
$= \int \tan(x)(\sec^2(x)-1)dx$
$= \int \tan(x)\sec^2(x)dx - \int \tan(x)dx$
*Since we learn from previous example that* $\int \tan(x)dx = \ln|\sec(x)| + c$*, we only need to find the result for the first term. Now, let's use substitution* $a = \tan(x)$*, then* $da = \sec^2(x)dx$ *so*

$$\int \tan(x)\sec^2(x)dx = \int udu$$

$= a^2 da = \frac{a^2}{2} = \frac{\tan^2(x)}{2}$ *since* $a = \tan(x)$
*Putting everything together, our final result is:*

$$\int \tan^3(x)dx = \frac{\tan^2(x)}{2} - \ln|\sec(x)| + C$$

*where* $C$ *is some constant.*

**Example 20.** *Evaluate* $\int \sec(x)dx$

$$\int \sec(x)dx$$

$= \int \frac{\sec(x)(\sec(x)+\tan(x))}{\sec(x)+\tan(x)}dx$
$= \int \frac{\sec^2(x)+\tan(x)\sec(x)}{\sec(x)+\tan(x)}dx$
*Now, let's* $v = \sec(x)+\tan(x)$*, then* $dv = \tan(x)\sec(x) + \sec^2(x)dx$ *so*

$$\int \sec(x)dx = \int \frac{1}{v}dv$$
$$= \ln|v| + c = \ln|\sec(x)+\tan(x)| + c$$

*where* $c$ *is some constant.*

**Example 21.** *Evaluate* $\int \frac{\sin^7(x)}{\cos^4(x)}dx$

$$\int \frac{\sin^7(x)}{\cos^4(x)}dx$$

$= \int \frac{\sin^6(x)}{\cos^4(x)}\sin(x)dx$
$= \int \frac{(\sin^2(x))^3}{\cos^4(x)}\sin(x)dx$
$= \int \frac{(1-\cos^2(x))^3}{\cos^4(x)}\sin(x)dx$
*Let's use substitution of* $b = \cos(x)$*, then* $db = -\sin(x)dx$

$$\int \frac{\sin^7(x)}{\cos^4(x)}dx = -\int \frac{(1-b^2)^3}{b^4}db$$

$= -\int \frac{(1-3b^2+3(b^2)^2-(b^2)^3)}{b^4}db$
$= -\int (b^{-4} - 3b^{-2} + 3 - b^2)db$
$= -\left(-\frac{1}{3b^3} + \frac{3}{b} + 3b - \frac{b^3}{3}\right) + c$ *and finally,*

$$\int \frac{\sin^7(x)}{\cos^4(x)}dx = \frac{1}{3\cos^3(x)} - \frac{3}{\cos(x)} - 3\cos(x) + \frac{\cos^3(x)}{3} + c$$

*where* $c$ *is some constant.*

**Example 22.** *Evaluate* $\int \frac{(x+2)}{\sqrt[3]{x-3}}dx$
*Let* $u = \sqrt[3]{x-3}$, *then* $x = u^3 + 3$ *and* $dx = 3u^2du$ *so*

$$\int \frac{(x+2)}{\sqrt[3]{x-3}}dx$$

$= \int \frac{(u^3+3)+2}{u}3u^2du$
$= \int (3u^4 + 15u)du$
$= \frac{3u^5}{5} + \frac{15u^2}{2} + c$ *and finally,*

$$\int \frac{(x+2)}{\sqrt[3]{x-3}}dx = \frac{3(x-3)^{\frac{5}{3}}}{5} + \frac{15(x-3)^{\frac{2}{3}}}{2} + c$$

*where c is some constant.*

**Theorem 3.** ***Integration by Trigonometry Substitution***
*Let* $a, b$ *be some constant, for some specific form of integrand, we usually have some specific trigonometry substitution.*

$$\sqrt{a^2 - b^2x^2} \overset{use}{\to} x = \frac{a}{b}\sin\theta \tag{4.1}$$

$$\sqrt{b^2x^2 - a^2} \overset{use}{\to} x = \frac{a}{b}\sec\theta \tag{4.2}$$

$$\sqrt{a^2 + b^2x^2} \overset{use}{\to} x = \frac{a}{b}\tan\theta \tag{4.3}$$

This is by far the hardest integration method among many well known ones. Many students are struggling with this because to tackle this method well, we need to be knowing very well on trigonometry and practising consistently on different exercises. Beside, some problems can be solved using this method but not letting $x$=something. Let's do a tricky example to see this point.

**Example 23.** *Evaluate* $\int e^{4x}\sqrt{1+e^{2x}}dx$
*Since the integrand has the from* (4.1) $\sqrt{a^2 + b^2x^2}$, *generally, we let* $x = \frac{a}{b}\tan\theta$. *However, if we do so, our problem becomes more complicated. This problem is tricky but we can still apply trigonometry substitution concept here.*
*If we let* $e^x = \tan\theta$, *we will have a good start.*
*Then,* $e^xdx = \sec^2\theta d\theta$
*and* $\sqrt{1+e^{2x}} = \sqrt{1+(e^x)^2} = \sqrt{1+\tan^2\theta} = \sqrt{\sec^2\theta} = |\sec\theta| = \sec\theta$
*(note that abslute value can be dropped since we are determining the indefinite integral). Therefore*

$$\int e^{4x}\sqrt{1+e^{2x}}dx = \int e^{3x}e^x\sqrt{1+e^{2x}}dx$$

$$= \int (e^x)^3\sqrt{1+e^{2x}}(e^x)dx = \int \tan^3\theta(\sec\theta)(\sec^2\theta)d\theta$$

*Now, this seems complicated still and we need to employ the method of U-substitution here as well (not trigonometry substitution), let's try* $u = \sec\theta$, *then* $du = \sec\theta\tan\theta d\theta$ *so we have:*

$$\int \tan^3\theta(\sec\theta)(\sec^2\theta)d\theta = \int (u^4 - u^2)du$$

$= \frac{u^5}{5} - \frac{u^3}{3} + c$ *where c is some constant*
$= \frac{\sec^5\theta}{5} - \frac{\sec^3\theta}{3} + c$
*We are not finished here yet because our answer is not in term of* $x$ *so we need to convert this back in term of* $x$*. Let's determine the relationship between* $\theta$ *and* $x$ *and by drawing a right triangle using the information of* $e^x = \tan\theta$*, we will have 2 legs equal to* $1$ *and* $e^x$ *and hypothenuse equal to* $\sqrt{1+e^{2x}}$ *with* $\theta$ *be adjacent to leg with length* $1$ *and hypothenuse. Then* $\tan\theta = \frac{e^x}{1}$ *and* $\sec\theta = \frac{\sqrt{1+e^{2x}}}{1} = \sqrt{1+e^{2x}}$ *so*
$\frac{\sec^5\theta}{5} - \frac{\sec^3\theta}{3} + c = \frac{(1+e^{2x})^{\frac{5}{2}}}{5} - \frac{(1+e^{2x})^{\frac{3}{2}}}{3} + c$ *and finally, we obtain:*

$$\int e^{4x}\sqrt{1+e^{2x}}dx = \frac{(1+e^{2x})^{\frac{5}{2}}}{5} - \frac{(1+e^{2x})^{\frac{3}{2}}}{3} + c$$

**Theorem 4.** ***Integration By Partial Fractions***
*Any rational function of a real variable can be written as the sum of a polynomial function and a finite number of partial fractions.*

Therefore, if the integrand is some rational function, we can break it down and express it as the sum of polynomial functions which has finite number of simpler terms so that we can integrate more easily.

This method basically take a big advantage of the property of linearity from integration.

In computing the integrals, this method mainly involves algebraic manipulation of the integrand and once we express it as partial fractions, the integration will be straight forward.

Without further ado, the table below is very helpful:
Let a rational function $f(x) = \frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials and the degree of $P(x)$ is smaller than the degree of $Q(x)$. Also, let $a, b, c \in R$ and $k \in N$ be some constants of the polynomial $Q(x)$ and $A_i, B_i$ with index $i$ be some coefficients of the partial fractions.

| $\mathbf{Q(x)}$ | **Form of the Partial Fraction** |
|---|---|
| $ax+b$ | $\frac{A}{ax+b}$ |
| $(ax+b)^k$ | $\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + ... + \frac{A_k}{(ax+b)^k}$ |
| $ax^2+bx+c$ | $\frac{Ax+B}{ax^2+bx+c}$ |
| $(ax^2+bx+c)^k$ | $\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + ... + \frac{A_kx+B_k}{(ax^2+bx+c)^k}$ |

**Summary 2.** ***Integration***
*Even though we can integrate the elementary functions without much trouble, there are many interesting complicated functions that we have difficulties with computing their integrals and none of our known method will work. For example, computing the integral of the normal function (the normal curve that is used in probability) goes beyond our known methods of integration. To solve those problems, we employ the technique called numberical analysis in which we only approximate the answer and not have the*

*exact solution.*

*Also, Matlab is one of the popular softwares that many mathematicians and scientists use for numberical analysis.*

*Furthermore, these methods introduced above are still very helpful for us to solve the integrals that involve roots, quadratics, trignometry functions and rational functions...*

*One big advantage of using these methods is that we get the exact solutions.*

*Beside, for some integrals, sometimes we use a mixture of many methods above to arrive the solutions and that kind of integration is usually tricky.*

### 4.5. **Fundamentals of Calculus.**

#### 4.5.1. ***First Part.***

**Theorem 5.** *Let $f(x)$ be a continuous real-valued function defined on $[a, b]$ and $F(x)$ defined in $[a, b]$ by*

$$F(x) = \int_a^x f(t)dt$$

*Then, $F(x)$ is continuous on $[a, b]$ and differtiable on $(a, b)$ and*

$$\mathbf{F'(x) = f(x)}$$

*for all $x \in (a, b)$.*

The first part merely introduces the idea that integration is the reverse process of differentiation as integration is to find antiderivative and differentiation is to find derivative. Furthermore, loosely speaking, the antiderivative of the derivative of a function is just the function itself. Interestingly, loosely speaking, the derivative of the antiderivative of a function is just the function itself, too.

Let $f(x) = F'(x)$, then $F(x) = \int_a^x f(t)dt + F(a)$ and

$$\mathbf{f(x) = \frac{d}{dx} \int_a^x f(t)dt}$$

#### 4.5.2. ***Second Part.***

**Theorem 6.** *Let $f(x)$ be continuous on $[a, b]$, and let $F(x)$ be an antiderivative such that $F(x) = \int f(x)dx$, then*

$$\mathbf{\int_a^b f(x)dx = F(b) - F(a)}$$

The second part introduces the computational aspect of integration. Determining the value of the definite integral is computed by calculating the difference between antiderivative evaluated at end point $b$ and the begining point $a$ respectively.

### 4.6. Sequence.

**Definition 8.** *A sequence is a* list of numbers *specified in a certain way by some function or it can also be a organized in a random way without any explicit function formula. The notations for sequence can be the following:* $\{a_1, a_2, a_3 ... a_n, a_{n+1} ...\}$, $\{a_n\}$ *or* $\{a_n\}_{n=1}^{\infty}$.
*Also, infinite sequence does not have an ending index n as n approches infinity and finite sequence is the one with finite ending index n.*

### 4.7. Series.

**Definition 9.** *A series is simply a* summation of a sequence. *We denote it as*
$a_1 + a_2 + a_3 + ... + a_n = \sum_{n=1}^{i=n} a_i$.

All finite series will be finite because we can always add up all finite number of terms.

However, since series is a summation of a list of numbers, will all infinite series be infinity? NO, not always! When we are trying to answer this question, we are determining whether or not the series is convergent (the answer is finite) or divergent (the answer is infinity).

There are many methods of testing convergency of the series including, but not limited to, integral test, comparison test, limit comparison test, alternating series test, ratio test, root test etc.

All these methods have its own advantages and disadvantages in determining the convergency of the series. Nevertheless, if the series is truly convergent, we should get the same answer no matter what testing method we use. If we use two or more tests and the answers are contradicting one another, we must be computing wrong in at least one testing method.

Since this is a review, I won't be going into too much details on each test but I will briefly go through the main substances of them.

#### 4.7.1. *Integral Test.*

**Theorem 7.** *Suppose that* $f(x)$ *is a positive, decreasing function on the interval* $[k, \infty)$ *and that* $f(n) = a_n$, *then*
*1. If* $\int_k^\infty f(x)dx$ *is convergent so is* $\sum_{n=k}^\infty a_n$.
*2. If* $\int_k^\infty f(x)dx$ *is divergent so is* $\sum_{n=k}^\infty a_n$.

Informally speaking, in this method, we view the sequence as some function and we take the integration of that function with the limit similar to the series index (some k as lower limit and infinity as the upper limit) and if the integral converges, we can conclude that the series also converges. Of course, there must a proof for this but since it is not short so we skip it here.

#### 4.7.2. *Comparison Test.*

**Theorem 8.** *Suppose that we have two series* $\sum a_n$ *and* $\sum b_n$ *with* $a_n$,$b_n \geq 0$ *for all n and* $a_n \leq b_n$ *for al n, then*
*1. If* $\sum b_n$ *is convergent then so is* $\sum a_n$.
*2. If* $\sum a_n$ *is divergent then so is* $\sum b_n$.

In this method, we start out with a series that is known to be either convergent or divergent (we usually use known P-Series which is a special type of series that we will review later), and we compare our current series with this known series. We have 3 main cases:
1. If the known series is divergent, and our series is greater than this for all index n, then our series also diverges
2. If the known series is convergent, and our series is less than this for all index n, then our series also converges.
3. For all other cases, we do not have a solution.

We again skip the proof here but the idea is simple. The convergent series solution is a finite number and if our series is smaller than this, our series solution must also be a finite number. Also, the divergent series solution is infinity and our series is bigger than that series for all n, our series solution, therefore, is not a finite number so it diverges.

### 4.7.3. ***Limit Comparison Test***.

**Theorem 9.** *Suppose that we have 2 series* $\sum a_n$ *and* $\sum b_n$ *with* $a_n$,$b_n \geq 0$ *for all* $n$. *Define,*

$$c = \lim_{n\to\infty} \frac{a_n}{b_n}$$

*If c is positive and is finite, then either both series converge or both series diverge.*

### 4.7.4. ***Alternating Series Test***.

**Theorem 10.** *Suppose that we have a series* $\sum a_n$ *and either* $a_n = (-1)^n b_n$ *or* $a_n = (-1)^{n+1} b_n$ *where* $b_n \geq 0$ *and if,*
*1.* $\lim_{n\to\infty} b_n = 0$ *and,*
*2.* $\{b_n\}$ *is eventually a decreasing sequence, mathmatically* $b_n \geq b_{n+1}$ $\forall n$
*then series* $\sum a_n$ *converges.*

when we use this test, this leads to the idea of absolute convergence.

**Definition 10.** *A series* $\sum a_n$ *is* ***absolutely convergent*** *if* $\sum |a_n|$ *is convergent. If* $\sum a_n$ *is convergent and* $\sum |a_n|$ *is divergent, then the series is* ***conditionally convergent***.

**Definition 11.** *If* $\sum a_n$ *is absolutely convergent, then it is also convergent.*

**Example 24.** $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$
*Using alternating series test, we have* $b_n = \frac{1}{n}$ *and we check:*
*1.* $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0$ *passed.*
*2.* $b_n = \frac{1}{n} > \frac{1}{n+1} = b_{n+1}$ $\forall n$ *passed. Refer to the proof below.*
*Therefore, we conclude that the series* $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n}$ *converges.*
*However, applying the idea of absolute convergence, we have*
$\sum_{n=1}^{\infty} |\frac{\cos(n\pi)}{n}| = \sum_{n=1}^{\infty} |\frac{(-1)^n}{n}| = \sum_{n=1}^{\infty} \frac{1}{n}$ *which is a divergent P-series with* $p = 1$.
*Therefore,* $\sum |a_n|$ *is divergent but* $\sum a_n$ *is convergent by alternating series test earlier so the series* $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n}$ *is* ***conditionally convergent***.

*Proof.* This proof is for the second condition $b_n = \frac{1}{n} > \frac{1}{n+1} = b_{n+1}\ \forall n$
For $n = 1$, $\frac{1}{1} > \frac{1}{2}$
For $n = 2$, $\frac{1}{2} > \frac{1}{3}$
...
Assume that $\frac{1}{n} > \frac{1}{n+1}\ \forall n \leq K$, show that it is also true for the case where $n = K + 1$.
For $n = K + 1$, we have $\frac{1}{K+1} > \frac{1}{(K+1)+1}$ (from our assumption) but $\frac{1}{(K+1)+1} = \frac{1}{K+2}$ so $\frac{1}{K+1} > \frac{1}{K+2}$.
By induction, $b_n = \frac{1}{n} > \frac{1}{n+1} = b_{n+1}\ \forall n$ □

### 4.7.5. ***Ratio Test.***

**Theorem 11.** *Suppose we have the series* $\sum a_n$. *Define* $L = \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}|$ *then*
*1. If* $L < 1$, *the series is absolutely convergent (convergent).*
*2. If* $L > 1$, *the series is divergent.*
*3. If* $L = 1$, *no conclusion.*

### 4.7.6. ***Root Test.***

**Theorem 12.** *Suppose that we have the series* $\sum a_n$. *Define,* $L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} |a_n|^{\frac{1}{n}}$, *then*
*1. If* $L < 1$, *the series is absolutely convergent (convergent).*
*2. If* $L > 1$, *the series is divergent.*
*3. If* $L = 1$, *no conclusion.*

### 4.7.7. ***Divergence Test.***

**Theorem 13.** *If* $\lim_{n\to\infty} a_n \neq 0$, *then* $\sum a_n$ *will diverge.*

This is the most basic and straight forward test and it should be applied first before we use the other ones. This is also referred as **divergence theorem**. As a compliment to it, we also have **convergence theorem**.

### 4.7.8. ***Convergence Theorem.***

**Theorem 14.** *If* $\sum a_n$ *converges, then* $\lim_{n\to\infty} a_n = 0$.

The reverse of this is not neccesarilly true, however.

## 4.8. **Special Convergent Series.**

### 4.8.1. ***Geometric Series.***

**Theorem 15.**

$$\mathbf{ar^0 + ar + ar^2 + ar^3 + ... = \sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}}$$

*provided that* $|r| < 1$ *which is referred as the radius of convergence.*

Let's prove it.

*Proof.* First, we have to show that $a1 + ar + ar^2 + ... + ar^n = a\left(\frac{1-r^{n+1}}{1-r}\right)$. We have $a1 + ar + ar^2 + ... + ar^n = a(1 + r + r^2 + ... + r^n)$ so we only need to show $1 + r + r^2 + ... + r^n = \frac{1-r^{n+1}}{1-r}$.
Using induction, when $n = 0$, we have $r^0 = \frac{1-r^0}{1-r} = \frac{1-r}{1-r} = 1$. Passed!
when $n = 1$, we have $r^0 + r^1 = 1 + r = \frac{1-r^2}{1-r} = \frac{(1-r)(1+r)}{1-r} = 1 + r$. Passed!
...
Assume that it is true for all $n \leq K$ which implies that $1+r+r^2+...+r^K = \frac{1-r^{(K+1)}}{1-r}$, show that it is also true when $n = K+1$. In another words, show that $1 + r + r^2 + ...r^K + r^{(K+1)} = \frac{1-r^{(K+2)}}{1-r}$.
We have $1+r+r^2+...r^K+r^{(K+1)} = \frac{1-r^{(K+1)}}{1-r} + r^{(K+1)}$ (from assumption) $= \frac{1-r^{(K+1)}+r^{(K+1)}(1-r)}{1-r} = \frac{1-r^{(K+1)}+r^{(K+1)}-r^{(K+2)}}{1-r} = \frac{1-r^{(K+2)}}{1-r}$.
Therefore, $1 + r + r^2 + ... + r^n = \frac{1-r^{n+1}}{1-r}$ $\forall n$.
Next,$ar^0 + ar + ar^2 + .... = \lim_{n\to\infty} \sum_{n=0}^{\infty} a\left(\frac{1-r^{n+1}}{1-r}\right) = a\left(\frac{1-0}{1-r}\right) = \frac{a}{1-r}$ provided that $|r| < 1$ since $r^\infty = 0$ if $|r| < 1$. ☐

### 4.8.2. ***P-Series.***

**Definition 12.** *The P-series is*

$$\sum_{\mathbf{n=1}}^{\infty} \frac{\mathbf{1}}{\mathbf{n^p}} = \frac{\mathbf{1}}{\mathbf{1^p}} + \frac{\mathbf{1}}{\mathbf{2^p}} + \frac{\mathbf{1}}{\mathbf{3^p}} + ...$$

*for any real number p.*

**Theorem 16.** *This series conveges only when $p > 1$, otherwise, it is divergent. In summary we have 3 cases:*

(a) $P < 1$, *the series diverges.*

(b) $P = 1$, *the series diverges and referred as* ***Harmonic Series***

(c) $P > 1$, *the series converges and referred as* ***Over-Harmonic Series***

## 4.9. Power Series.

**Definition 13.** *A power series in x about c with coefficients $a_n$ is an infinite series such that*

$$\sum_{\mathbf{n=0}}^{\infty} \mathbf{a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2} + ...$$

Geometric series is a special case of power series with $c = 0$ and fixed coefficients $a_n = a$ $\forall n$ such that

$$\sum_{n=1}^{\infty} ax^{n-1} = ax^0 + ax + ax^2 + ...$$

**Theorem 17.** *For a convergent power series $\sum_{n=0}^{\infty} a_n(x-c)^n$, we have:*

(a) *It converges for all x.*
(b) *It converges for all $x \in (c - R_1, c + R_1)$ around c but diverges outside outside $[c-R_1, c+R_1]$ and $R_1$ is referred as the radius of convergence.*
(c) *It converges only for $x = c$.*

Beside, we have 2 more essential theorems regarding power series below.

**Theorem 18.** ***Integration of Power Series***
*Let* $f(x)$ *be defined by a power series* $\sum_{n=0}^{\infty} a_n(x-c)^n$ *on its interval of convergence (with radius convergence of* $R_1$*), then:*

$$\int \mathbf{f(x)dx} = \sum_{\mathbf{n=0}}^{\infty} \mathbf{a_n} \frac{\mathbf{(x-c)^{n+1}}}{\mathbf{(n+1)}} + \mathbf{K}$$

*for* $|x-c| < R_1$*.*

**Theorem 19.** ***Differentiation of Power Series***
*Let* $f(x)$ *be defined by a power series* $\sum_{n=0}^{\infty} a_n(x-c)^n$ *on its interval of convergence (with radius convergence of* $R_1$*), then* $f(x)$ *is differentiable in that interval and*

$$\mathbf{f'(x)} = \sum_{\mathbf{n=0}}^{\infty} \mathbf{na_n(x-c)^{n-1}}$$

*for* $|x-c| < R_1$*.*

### 4.10. **Taylor and Maclaurin Series.**

**Definition 14.** ***Talor Series***
*Let* $f(x)$ *be a function that is infinitely differentiable at* $x = c$*, in another words, the derivatives* $f^{(n)}(c)$ *exist for all positive integers* $n$*, then the Taylor Series for* $f(x)$ *about* $c$ *is the power series*
$\sum_{\mathbf{n=0}}^{\infty} \mathbf{a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2} + ...$ *with*

$$\mathbf{a_n} = \frac{\mathbf{f^{(n)}(c)}}{\mathbf{n!}}$$

*for all* $n$*.*

Comment: $f^{(0)}(x) = f(x) \Rightarrow a_0 = f(c)$.

**Definition 15.** ***Maclaurin Series***
*The Maclaurin Series for* $f(x)$ *is the Taylor Series for* $f(x)$ *about* $0$ *so*
$\sum_{\mathbf{n=0}}^{\infty} \mathbf{a_n(x-0)^n = a_0 + a_1(x-0) + a_2(x-0)^2} + ... = \mathbf{a_0 + a_1x + a_2x^2} + ...$
*where*

$$\mathbf{a_n} = \frac{\mathbf{f^{(n)}(c)}}{\mathbf{n!}}$$

*for all* $n$*.*

This looks like geometric series but it is NOT geometric series because the coefficients $a_n$ in Maclaurin series is NOT fixed with for geometric series, we have $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + ...$ where $a_n = a \; \forall n$.

As a results of Taylor Series, many elementary functions can be represented by a power series and many of these can easily be verified using the theorem above.

#### 4.10.1. ***Exponential function.***

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3} + ... \tag{4.4}$$

for all $x$

4.10.2. ***Natural logarithm.***

$$ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n} \tag{4.5}$$

for $|x| \leq 1, x \neq 1$

4.10.3. ***Finite geometric series.***

$$\frac{1-x^{m+1}}{1-x} = \sum_{0}^{m} x^n \tag{4.6}$$

for $x \neq 1$ and $m \in N_0$

4.10.4. ***Infinite geometric series.***

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \tag{4.7}$$

for $|x| < 1$

4.10.5. ***Variants of infinite geometric series.***

$$\frac{x^m}{1-x} = \sum_{n=m}^{\infty} x^n \tag{4.8}$$

for $|x| < 1$ and $m \in N_0$

$$\frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} nx^n \tag{4.9}$$

for $|x| < 1$

4.10.6. ***Square root function.***

$$\sqrt{1+x} = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{(1-2n)(n!)^2 4^n} x^n \tag{4.10}$$

for $|x| < 1$

4.10.7. ***Binomial series.***

$$(1+x)^{\alpha} = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n \tag{4.11}$$

for all $|x| < 1$ and all complex $\alpha$

### 4.10.8. ***Trigonometric functions.***

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ... \tag{4.12}$$

for all $x$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ... \tag{4.13}$$

for all $x$

$$\sinh(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + ... \tag{4.14}$$

for all $x$

$$\cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + ... \tag{4.15}$$

for all $x$ Beside, based on these known power series, we can use them to determine the coefficients of the power series of more complicated functions that involve the mixture of these elementary functions.

**Summary 3.** ***Calculus***

*Calculus is one of the main tools exclusively used in almost all science and engineering disciplines from solving a mechanical physical problem to estimating the population in ecology. The reason is that algebra alone is not sufficient in dealing with those complex problems in science and engineering.*

*That is why Leibniz and Isaac Newton gathered many ideas from many different sources and create this coherent calculus and apply it to solve physics problems. In addition, calculus was widely spreaded to other areas and not suprisingly, many scientists and engineers adopted it as soon as they found it useful.*

*Calculus is also the starndard mathematics that is widely taught in almost all universities worldwide and all engineering, science and math majors have calculus in their educational curricula.*

*We have no doubt at all calculus is such a powerful tool in science and engineering societies and calculus applications are endless. Nowaday, not only is calculus applied to science and engineering alone, it is increasingly used in finance, actuary, and business etc*

*For science believers, there is not any reason at all not to master calculus! Learning calculus is both fun and interesting. Many technologies today like electricity, cell phones, computer programs, computer hardwares, cars... would not even exist without calculus.*

## 5. Fourier Series

### 5.1. Introduction to Fourier Series.

**Definition 16.** *Fourier series is a trigonometric series that is used to approximate or even express any real periodic function $f(x)$ with period $T$*

*and it is defined as:*

$$\mathbf{f(x)} = \sum_{\mathbf{n=0}}^{\infty} \left[\mathbf{A_n} \cos\left(\frac{\mathbf{2\pi nx}}{\mathbf{T}}\right) + \mathbf{B_n} \sin\left(\frac{\mathbf{2\pi nx}}{\mathbf{T}}\right)\right]$$

*where*

$$A_n = \frac{2}{T}\int_0^T f(x)\cos\left(\frac{2\pi nx}{T}\right)dx$$
$$B_n = \frac{2}{T}\int_0^T f(x)\sin\left(\frac{2\pi nx}{T}\right)dx$$

Notes: When $n = 0$, we have $A_0 = \frac{2}{T}\int_0^T f(x)dx$ since $\cos(0) = 1$ and $B_0 = 0$ since $\sin(0) = 0$.

**Theorem 20.** ***Special Fourier Series with a period of*** $2\pi$

$$\mathbf{f(x)} = \frac{\mathbf{a_0}}{\mathbf{2}} + \sum_{\mathbf{n=1}}^{\infty} [\mathbf{a_n}\cos(\mathbf{nx}) + \mathbf{b_n}\sin(\mathbf{nx})]$$

*where*

$$a_0 = \frac{2}{\pi}\int_{-\pi}^{\pi} f(x)dx$$
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx$$

Notes: In this case, when $n = 0$, they have the first term $\frac{a_0}{2}$ (the $a_0$ here is not true $a_0$) but this term itself $\frac{a_0}{2}$ is the true $a_0$ since
revised $a_0 = \frac{2}{\pi}\int_{-\pi}^{\pi} f(x)dx$ but the true $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$.
This is, however, the standard way to express a Fourier Series for $T = 2\pi$.

Now, we will go through the derivations of the coefficients $a_n$ and $b_n$ of the Fourier Series with period of $2\pi$ but before doing so, we need to explore **some other math concepts** to achieve our objective.

**Definition 17.** ***Sequence of Orthogonal functions***
*A sequence of functions is* ***orthogonal*** *if*

$$\int_a^b \theta_m(x)\theta_n(x)dx = \begin{cases} 0 & m \neq n \\ \neq 0 & m = n \end{cases}$$

**Definition 18.** ***Sequence of Orthonormal functions***
*A sequence of functions is* ***orthonormal*** *if*

$$\int_a^b \theta_m(x)\theta_n(x)dx = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

**Definition 19.** ***Length of functions***
*Let $f(x)$ be a real function with single variable $x$, the length of function $f(x)$ in the interval $[a, b]$ is defined as:*

$$||f(x)|| = \sqrt{(f, f)} = \sqrt{\int_a^b f^2(x)dx}$$

The "‘length"’ in this context does not refer to the physical length but the idea here is very close to the the length of vector. In another words, this length is viewed as the magnitude of the function. A null function defined below has length of 0 which implies that the null function is essentially 0 but it doesn’t neccessarily mean 0 everywhere so the null function can be discontinuous function. However, null function can also be 0 everywhere but it not always have to behave like this.

**Definition 20.** ***Null function***
*Function $f(x)$ is called a null function if $||f(x)|| = 0$*

**Theorem 21.** ***Distance between two functions***
*Let $f(x)$ and $g(x)$ be two real single variable functions, then the distance between the two functions in the interval $[a, b]$ is defined as:*

$$d(f, g) = ||f(x) - g(x)|| = \sqrt{\int_a^b (f(x) - g(x))^2 dx}$$

**Theorem 22.** ***Mean Square Error of the Approximation***
*Let $f(x)$ be a real function and series $P_n(x)$ be an approximation of $f(x)$ in the interval $[a, b]$, then the mean square error of the approximation is $E_n$:*

$$E_n = d^2(f(x), P_n(x)) = ||f(x), P_n(x)||^2 = \int_a^b (P_n(x) - f(x))^2 dx$$

**Theorem 23.** ***Convergence of approximation to function***
*Let $f(x)$ be a real function and series $P_n(x)$ be an approximation of $f(x)$*

(a) ***Pointwise Convergence***
*$P_n(x)$ converges pointwise to function $f(x)$ if pointwise*

$$\lim_{n \to \infty} f_n(x) = f(x)$$

(b) ***Convergence In The Mean***
*$P_n(x)$ converges in the mean to function $f(x)$ if mean square error of the approximation approaches 0. In another words,*

$$\lim_{n \to \infty} E_n = \lim_{n \to \infty} d^2(f(x), P_n(x)) = \lim_{n \to \infty} ||f(x), P_n(x)||^2 = \lim_{n \to \infty} \int_a^b (P_n(x) - f(x))^2 dx = 0$$

### 5.2. **Derivation of Fourier Coefficients.**

**Theorem 24.** *Let $\{\phi_n\}$ be a sequence of orthonormal functions such as $\{\frac{1}{2\pi}, \frac{\cos(x)}{\sqrt{\pi}}, \frac{\sin(x)}{\sqrt{\pi}}, \frac{\cos(2x)}{\sqrt{\pi}}, \frac{\sin(2x)}{\sqrt{\pi}}, ...\}$, let $f(x)$ be a real function on $[a, b]$ such that $\int_a^b f^2(x)dx < \infty$ and define series $P_n$ as*

$$P_n(x) = c_1\phi_1(x) + c_2\phi_2(x) + c_3\phi_3(x) + ... + c_n\phi_n(x)$$

*Then,* ***Fourier coefficients minimize the mean square error*** *between $f(x)$ and $P_n(x)$ and*

$$\min \mathbf{E_n} = \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{f(x)^2dx} - \sum_{\mathbf{k=1}}^{\mathbf{n}} \mathbf{(a_k)^2}$$

*where*

$$\mathbf{a_k} = \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{f(x)\phi_k(x)dx}$$

*Proof.* Mean square error of the approximation$P_n(x)$ of $f(x)$ is:

$$\int_a^b [f(x) - P_n(x)]^2 dx$$

$$= \int_a^b f^2(x)dx - 2\int_a^b f(x)P_n(x)dx + \int_a^b P_n^2(x)dx$$

We break the original integrals into 3 smaller integrals.
Since $P_n(x) = c_1\phi_1(x)+c_2\phi_2(x)+c_3\phi_3(x)+...+c_n\phi_n(x)$ the second integral

$$2\int_a^b f(x)P_n(x)dx$$

$$= 2\int_a^b f(x)[c_1\phi_1(x) + c_2\phi_2(x) + c_3\phi_3(x) + ... + c_n\phi_n(x)]dx$$

$$= 2\left(c_1\int_a^b f(x)\phi_1(x)dx + c_2\int_a^b f(x)\phi_2(x)dx + ... + c_n\int_a^b f(x)\phi_n(x)dx\right)$$

Now let

$$a_k = \int_a^b f(x)\phi_k(x)dx$$

so

$$\mathbf{2\int_a^b f(x)P_n(x)dx = 2\sum_{k=1}^{n} c_k a_k}$$

In addition, the third integral

$$\int_a^b P_n^2(x)dx$$

$$= \int_a^b [c_1\phi_1(x) + c_2\phi_2(x) + c_3\phi_3(x) + ... + c_n\phi_n(x)][c_1\phi_1(x) + c_2\phi_2(x) + c_3\phi_3(x) + ... + c_n\phi_n(x)]dx$$

$$= \int_a^b c_1^2\phi_1^2 dx + \int_a^b c_1\phi_1 b_2\phi_2 dx + \int_a^b c_1\phi_1 b_3\phi_3 dx + \int_a^b c_1\phi_1 b_4\phi_4 dx + ... + \int_a^b c_1\phi_1 b_n\phi_n dx$$

$$+ \int_a^b c_2\phi_2 c_1\phi_1 dx + \int_a^b c_2^2\phi_2^2 dx + \int_a^b c_2\phi_2 b_3\phi_3 dx + \int_a^b c_2\phi_2 b_4\phi_4 dx + ... + \int_a^b c_2\phi_2 b_n\phi_n dx$$

$$+ \int_a^b c_3\phi_3 c_1\phi_1 dx + \int_a^b c_3\phi_3 c_2\phi_2 dx + \int_a^b c_3^2\phi_3^2 dx + \int_a^b c_3\phi_3 c_4\phi_4 dx + ... + \int_a^b c_3\phi_3 c_n\phi_n dx$$

$$+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots +$$

$$+ \int_a^b c_n\phi_n c_1\phi_1 dx + \int_a^b c_n\phi_n c_2\phi_2 dx + \int_a^b c_n\phi_n c_3\phi_3 dx + \int_a^b c_n\phi_n c_4\phi_4 dx + ... + \int_a^b c_n^2\phi_n^2 dx$$

$$= c_1^2 + c_2^2 + c_3^2 + c_4^2 + ... + c_n^2 = \mathbf{\sum_{k=1}^{n} c_k^2 = \int_a^b P_n^2(x)dx}$$

since $\{\phi_n\}$ is an orthonormal function (i.e. $\int_a^b \phi_k^2 dx = 1$ and $\int_a^b \phi_m\phi_n dx = 0$ when $m \neq n$). Therefore,

$$\mathbf{\int_a^b [f(x) - P_n(x)]^2 dx = \int_a^b f^2(x)dx + \sum_{k=1}^{n} -2c_k a_k + c_k^2}$$

$$= \int_a^b f^2(x)dx + \sum_{k=1}^{n} -2c_k a_k + c_k^2 + a_k^2 - a_k^2 = \int_a^b f^2(x)dx + \sum_{k=1}^{n} a_k^2 - 2c_k a_k + c_k^2 + -a_k^2$$

$$= \int_a^b f^2(x)dx + \sum_{k=1}^{n} (a_k - c_k)^2 - \sum_{k=1}^{n} a_k^2$$

Hence, the mean square error is mininized when

$$\sum_{k=1}^{n} (a_k - c_k)^2 = 0$$

which implies that

$$a_k = c_k$$

but $a_k = \int_a^b f(x)\phi_k(x)dx$ so the minimizer is

$$c_k = a_k = \int_a^b f(x)\phi_k(x)dx$$

and this is the form for Fourier coefficients.
Also, the minimum is

$$\min E_n = \int_a^b f(x)^2 dx + 0 - \sum_{k=1}^{n} (a_k)^2$$

□

### 5.3. **Representation of Fourier Series.**

#### 5.3.1. ***Standard Representation for Fourier Series with period T.***

$$\mathbf{f(x)} = \sum_{\mathbf{n=0}}^{\infty} \left[\mathbf{A_n} \cos\left(\frac{\mathbf{2\pi nx}}{\mathbf{T}}\right) + \mathbf{B_n} \sin\left(\frac{\mathbf{2\pi nx}}{\mathbf{T}}\right)\right]$$

where

$$A_n = \frac{2}{T}\int_0^T f(x)\cos\left(\frac{2\pi nx}{T}\right)dx$$

$$B_n = \frac{2}{T}\int_0^T f(x)\sin\left(\frac{2\pi nx}{T}\right)dx$$

5.3.2. ***Representation with only Cosine for Fourier Series with period T.*** Let $C = \sqrt{A^2+B^2}$, then

$$f(x) = A\cos\left(\frac{2\pi nx}{T}\right) + B\sin\left(\frac{2\pi nx}{T}\right)$$

$$= \left[\frac{\sqrt{A^2+B^2}}{\sqrt{A^2+B^2}}\left(A\cos\left(\frac{2\pi nx}{T}\right) + B\sin\left(\frac{2\pi nx}{T}\right)\right)\right]$$

$$= C\left[\frac{A}{\sqrt{A^2+B^2}}\cos\left(\frac{2\pi nx}{T}\right) + \frac{B}{\sqrt{A^2+B^2}}\sin\left(\frac{2\pi nx}{T}\right)\right]$$

$$= C\left[\cos(\theta)\cos\left(\frac{2\pi nx}{T}\right) + \sin(\theta)\sin\left(\frac{2\pi nx}{T}\right)\right]$$

$$= C\left[\cos\left(\frac{2\pi nx}{T} - \theta\right)\right]$$

$$= \sqrt{A^2+B^2}\left[\cos\left(\frac{2\pi nx}{T} - \tan^{-1}\left(\frac{B}{A}\right)\right)\right]$$

Therefore, the general form for Fourier Series with cosine terms only is:

$$\mathbf{f(x)} = \sum_{\mathbf{n=0}}^{\infty} \mathbf{C_n}\left[\cos\left(\frac{\mathbf{2\pi nx}}{\mathbf{T}} - \theta\right)\right]$$

or

$$f(x) = \sqrt{A_n^2+B_n^2}\left[\cos\left(\frac{2\pi nx}{T} - \tan^{-1}\left(\frac{B}{A}\right)\right)\right]$$

5.3.3. ***Complex Exponential Representation for Fourier Series with period T***. We know that $e^{i\theta} = \cos\theta + i\sin\theta$ $\begin{cases}\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}\\ \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}\end{cases}$

$$f(x) = \sum_{n=0}^{\infty}\left[A_n\cos\left(\frac{2\pi nx}{T}\right) + B_n\sin\left(\frac{2\pi nx}{T}\right)\right]$$

and let

$$\omega = \frac{2\pi nx}{T}$$

then

$$f(x) = \sum_{n=0}^{\infty}\left[A_n\left(\frac{e^{i\omega}+e^{-i\omega}}{2}\right) + B_n\left(\frac{e^{i\omega}-e^{-i\omega}}{2i}\right)\right]$$

$$f(x) = \sum_{n=0}^{\infty}\left[\frac{A_ne^{i\omega}+A_ne^{-i\omega}}{2} + \frac{B_ne^{i\omega}}{2i} - \frac{B_ne^{-i\omega}}{2i}\right]$$

$$f(x) = \sum_{n=0}^{\infty}\left[e^{i\omega}\left(\frac{A_n}{2}+\frac{B_n}{2i}\right) + e^{-i\omega}\left(\frac{A_n}{2}-\frac{B_n}{2i}\right)\right]$$

$$f(x) = \sum_{n=0}^{\infty}\left[e^{i\omega}\left(\frac{A_n}{2}+\frac{B_n}{2i}\right)\left(\frac{i}{i}\right) + e^{-i\omega}\left(\frac{A_n}{2}-\frac{B_n}{2i}\right)\left(\frac{i}{i}\right)\right]$$

$$f(x) = \sum_{n=0}^{\infty}\left[e^{i\omega}\left(\frac{B_ni-A_n}{-2}\right) + e^{-i\omega}\left(\frac{-A_n-B_ni}{-2}\right)\right]$$

$$f(x) = \sum_{n=0}^{\infty}\left[e^{i\omega}\left(\frac{A_n-B_ni}{2}\right)\right] + \sum_{n=-\infty}^{0}\left[e^{i\omega}\left(\frac{A_n+B_ni}{2}\right)\right]$$

Therefore, the general form of complex exponential representation for Fourier Series with period T is:

$$\mathbf{f(x)} = \sum_{-\infty}^{\infty}\mathbf{C_n e^{i\omega}} = \sum_{-\infty}^{\infty}\mathbf{C_n e^{i\left(\frac{2\pi nx}{T}\right)}}$$

where $\begin{cases}\mathrm{C}_0 = A_0 & \text{n=0}\\ \mathrm{C}_n = \frac{A_n-B_ni}{2} & \forall n>0\\ \mathrm{C}_n = \frac{A_n+B_ni}{2} & \forall n<0\end{cases}$ Particularly, when $n > 0$, we have:

$$C_n = \frac{A_n-B_ni}{2}$$

$$= \frac{1}{T}\int_0^T f(x)\cos\left(\frac{2\pi nx}{T}\right)dx - \frac{i}{T}\int_0^T f(x)\sin\left(\frac{2\pi nx}{T}\right)dx$$

$$= \frac{1}{T}\int_0^T \left[f(x)\cos\left(\frac{2\pi nx}{T}\right) - i\sin\left(\frac{2\pi nx}{T}\right)\right]dx$$

$$= \mathbf{\frac{1}{T}\int_0^T f(x)e^{-i\left(\frac{2\pi nx}{T}\right)}dx}$$

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