What Variance Risk Premium Tells Us About the Expected Market Returns

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A Predictive Regression

\[ R_{m,t+1} = \beta_0 + \beta_p \text{Predictor}_t + \epsilon_{t+1} \]

- To predict a variable at time \( T \) (Traditional Approach)
  1) Run above regression for \( t = T - k \) to \( t = T - 1 \) and obtain \( \hat{\beta}_{0,T} \) and \( \hat{\beta}_{p,T} \)
  2) Form out-of-sample predictions
     \[ R_{m,T+1|T} = \hat{\beta}_{0,T} + \hat{\beta}_{p,T} \text{Predictor}_T \]

- Choosing \( k \) can be a problem when
  - \( R^2 \) is low \( \rightarrow \) requires high \( k \)
  - Predictive relation changes over time \( \rightarrow \) requires low \( k \)
Predictors of Market Returns

- Well known predictors (e.g. Dividend yield, Term Premium, Default Premium etc.)
  - Restricted to long term returns (requires long data)
  - In-sample $R^2$ low, out-of-sample $R^2$ lower, even negative

- The variance risk premium (VRP) predicts short-run returns (Bollerslev, Tauchen and Zhou 2009)
  - Low $R^2$
  - Question: Does the predictive relation change over time?
  - Time-varying risk aversion (Todorov, 2009), uncertainty (Drechsler and Yaron, 2011), negative jump risk (Bollerslev, Todorov and Xu 2015)
  - Neglected in the literature: price and variance move in the opposite direction.
Suppose:

\[ R_{m,t+1} = \alpha_c + \beta_c (RV_{t+1} - E[RV_{t+1}]) + \epsilon_{t+1} \]

Take \( E^Q[\cdot] - E[\cdot] \) from the above equation. Then, approximately,

\[
\underbrace{R_f - E[R_{m,t+1}]}_{-\text{MRP}} = \underbrace{\beta_c [E^Q[RV_{t+1}] - E[RV_{t+1}]]}_{\text{VRP}} + E^Q[\epsilon_{t+1}]
\]

H1. The predictive relation between the VRP and the market returns is determined by \( \beta_c \)

H2. The predictive power depends on \( Corr(R_{m,t+1}, RV_{t+1} - E[RV_{t+1}]) \)
Forecasting Monthly Returns

▷ Traditional Approach
  1) First stage: \[ R_{m,t+1} = \beta_0 + \beta_p VRP_t + \epsilon_{t+1} \]
     • From month \( t = T-k \) to \( t = T-1 \)
  2) OOS Prediction = \[ \hat{\beta}_0 + \hat{\beta}_p VRP_T \]

▷ Contemporaneous Beta Approach
  1) First stage: \[ R_{m,t} = \alpha_c - \beta_c (RV_t - E[RV_t]) + \epsilon_t \]
     • High-frequency data \( \rightarrow \) daily \( RV / \) returns during month \( T \)
  2) OOS Prediction = \[ R_f + \hat{\beta}_c VRP_T \]

→ Uses only short-term data and has higher \( R^2 \)
The Model

Assumption:

\[
\frac{dS_t}{S_t} = \mu_t dt + \sqrt{V_t} (\rho_t dW^v_t + \sqrt{1 - \rho_t^2} \sqrt{V_t} dW^o_t)
\]

\[
dV_t = \theta(V_t) dt + \sigma_{vt} dW^v_t
\]

This implies the following two-factor structure:

\[
\frac{dS_t}{S_t} = \left[ \mu_t + \rho_t \frac{\sqrt{V_t}}{\sigma_{vt}} (dV_t - \kappa(\theta - V_t)) \right] dt + \sqrt{1 - \rho_t^2} \sqrt{V_t} dW_t^o
\]  (1)

Taking the variance process under the EMM and subtracting the variance process under the P measure gives,

\[
VRP = \sigma_{vt} \lambda^v_t
\]

The drift of Equation (1) under the EMM is the risk-free rate. Solving these equations together,

\[
\mu_t - r = -\rho_t \frac{\sqrt{V_t}}{\sigma_v} VRP_t + \sqrt{1 - \rho_t^2} \sqrt{V_t} \lambda^o_t
\]
Estimation Methodologies

- **VRP**
  - VRP\(_N\) = \(\frac{VIX^2}{12}\) - Realized Variance
  - VRP\(_I\) = \(\frac{VIX^2}{12}\) - E[Integrated Variance]

- **Contemporaneous Betas**: Daily regressions one for each month
  \[
  R_{m,t} = \alpha_c - \beta_c (RV_t - E[RV_t]) + \epsilon_t
  \]

- **Contemporaneous Correlations**: monthly corr between daily values
  \[
  Corr(R_{m,t+1}, RV_{t+1} - E_t[RV_{t+1}])
  \]
Estimation Methodologies

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- **Contemporaneous Betas**: Daily regressions one for each month
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- **Contemporaneous Correlations**: monthly corr between daily values
  \[ \text{Corr}(R_{m,t+1}, RV_{t+1} - E_t[RV_{t+1}]) \]

- **Main Volatility Model**: HAR-RVAR (Corsi 2009)
  (also Vol version and RGARCH)
  \[ \sum_{j=1}^{k} RV_{\tau+j} = a_0 + a_d RV_{\tau} + a_w (\sum_{j=0}^{4} RV_{\tau-j}) + a_m (\sum_{j=0}^{21} RV_{\tau-j}) + \phi \tau + 1 \]
In-Sample Regressions (1) - Betas

Test the relation between the predictive and the contemporaneous betas

\[ R_{m,t+1} = \gamma_0 + \gamma_v V R P_t + \gamma_I \beta_{c,t} \times V R P_t + \epsilon_{t+1} \]

<table>
<thead>
<tr>
<th>Dep Var</th>
<th>One-month Predictive Market Returns</th>
</tr>
</thead>
<tbody>
<tr>
<td>( V R P_N )</td>
<td>0.045 0.047</td>
</tr>
<tr>
<td></td>
<td>(1.82) (1.89)</td>
</tr>
<tr>
<td>( V R P_I )</td>
<td>0.044 0.044</td>
</tr>
<tr>
<td></td>
<td>(3.20) (2.89)</td>
</tr>
<tr>
<td>( \text{VRP} \times \beta_c )</td>
<td>-0.679 -0.719</td>
</tr>
<tr>
<td></td>
<td>(-2.13) (-2.12)</td>
</tr>
<tr>
<td>Adj-( R^2 )</td>
<td>0.035 0.048 0.066 0.084</td>
</tr>
</tbody>
</table>
### In-Sample Regressions (2) - Correlations

<table>
<thead>
<tr>
<th></th>
<th>Months with Corr that are</th>
<th></th>
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<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>High</td>
<td>Medium</td>
<td>Low</td>
<td></td>
</tr>
<tr>
<td>VRP&lt;sub&gt;N&lt;/sub&gt;</td>
<td>In-sample $R^2$</td>
<td>0.185</td>
<td>0.151</td>
<td>0.004</td>
</tr>
<tr>
<td></td>
<td>Predictive beta ($\beta_p$)</td>
<td>0.139</td>
<td>0.087</td>
<td>-0.013</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(3.88)</td>
<td>(5.24)</td>
<td>(-0.36)</td>
</tr>
<tr>
<td>VRP&lt;sub&gt;I&lt;/sub&gt;</td>
<td>In-sample $R^2$</td>
<td>0.211</td>
<td>0.106</td>
<td>0.000</td>
</tr>
<tr>
<td></td>
<td>Predictive beta ($\beta_p$)</td>
<td>0.132</td>
<td>0.041</td>
<td>-0.002</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(3.92)</td>
<td>(5.01)</td>
<td>(-0.04)</td>
</tr>
<tr>
<td></td>
<td>Number of months</td>
<td>87</td>
<td>76</td>
<td>65</td>
</tr>
</tbody>
</table>

Predictions are more accurate when the correlations are higher.
Out-of-Sample Predictions (1) - Betas

\[ OOS - R^2 = 1 - \frac{\sum_t (\hat{R}_{m,t+1|t} - R_{m,t+1})^2}{\sum_t (R_{m,t} - R_{m,t+1})^2} \]

<table>
<thead>
<tr>
<th></th>
<th>Predictive Beta</th>
<th>Contep. Beta</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>5y rolling</td>
<td>10y rolling</td>
</tr>
<tr>
<td>VRP_N</td>
<td>OOS-R^2</td>
<td>-0.057</td>
</tr>
<tr>
<td></td>
<td>Wald Stat</td>
<td>0.608</td>
</tr>
<tr>
<td></td>
<td>P-value</td>
<td>(0.436)</td>
</tr>
<tr>
<td>VRP_I</td>
<td>OOS-R^2</td>
<td>-0.169</td>
</tr>
<tr>
<td></td>
<td>Wald Stat</td>
<td>0.697</td>
</tr>
<tr>
<td></td>
<td>P-value</td>
<td>(0.404)</td>
</tr>
</tbody>
</table>
## Out-of-Sample Predictions (2) - Correlations

<table>
<thead>
<tr>
<th>VRP</th>
<th>Betas</th>
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<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>High</td>
<td>Medium</td>
<td>Low</td>
<td>High-Low</td>
</tr>
<tr>
<td>VRP_N</td>
<td>5y Predictive</td>
<td>0.004</td>
<td>-0.037</td>
<td>-0.110</td>
<td>0.114</td>
<td></td>
</tr>
<tr>
<td></td>
<td>10y Predictive</td>
<td>0.138</td>
<td>0.106</td>
<td>-0.172</td>
<td>0.309</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Extended Sample</td>
<td>0.116</td>
<td>0.104</td>
<td>-0.148</td>
<td>0.264</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Contemp. Beta</td>
<td>0.129</td>
<td>0.021</td>
<td>0.072</td>
<td>0.057</td>
<td></td>
</tr>
<tr>
<td>VRP_I</td>
<td>5y Predictive</td>
<td>0.152</td>
<td>-1.113</td>
<td>-0.079</td>
<td>0.231</td>
<td></td>
</tr>
<tr>
<td></td>
<td>10y Predictive</td>
<td>0.231</td>
<td>-0.257</td>
<td>-0.074</td>
<td>0.305</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Extended Sample</td>
<td>0.183</td>
<td>-0.040</td>
<td>-0.051</td>
<td>0.234</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Contemp. Beta</td>
<td>0.149</td>
<td>0.042</td>
<td>0.080</td>
<td>0.070</td>
<td></td>
</tr>
<tr>
<td></td>
<td># of Months</td>
<td>57</td>
<td>60</td>
<td>63</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Conclusion

- The VRP and the market premium is related in a particular way
  - The contemporaneous relation between the price and variance explains why they are related
  - The contemporaneous beta is the slope that determines the predictive relation
  - Contemporaneous beta approach outperforms the traditional approach

- Possible extensions
  - What determines the market premium when the correlations are low?
  - How does dividend yield fit into this picture?