Problem 4.

Determine the deformation of a bar under its own weight. What is the equivalent load at the end of the bar that can replace the self-weight?

Consider the deformation of an element of length $dz$. The weight acting on it is

$$ P = \rho g A (L - z) $$

where $\rho =$ density of the bar; $A =$ cross-sectional area; $g =$ gravitational acceleration. Hence, the deformation of the element

$$ d\delta = \frac{P dz}{AE} = \frac{\rho g A (L - z) dy}{AE} $$

Total deformation of the bar

$$ \delta = \int_0^L \frac{\rho g}{E} (L - z) dz $$

$$ = \frac{\rho g}{E} \int_0^L (L - z) dz $$

$$ = \frac{\rho g}{E} \left( L^2 - \frac{L^2}{2} \right) $$

$$ = \frac{\rho g L^2}{2E} $$

Equivalent force at the end

$$ \sigma A $$

$$ = (E\epsilon) A $$

$$ = \frac{E}{L} \delta A $$

$$ = \frac{E \rho g L^2 A}{2EL} $$

$$ = \frac{1}{2} \frac{\rho g A L}{2} = \frac{W}{2} $$

where $W$ is the total weight of the bar.

Problem 5.

Determine the deformation at point C. Assume the bar ACD is rigid.

$$ \sum F_y = 0 $$

$$ \Rightarrow F_{AB} + F_{DE} = 45 \text{ kN} $$

$$ \sum M_D = 0 $$

$$ \Rightarrow -F_{AB} \cdot (0.6 \text{ m}) + (45 \text{ kN}) \cdot (0.4 \text{ m}) = 0 $$

$F_{AB} = 30 \text{ kN}$

$F_{DE} = 15 \text{ kN}$
\[
\delta_{AB} = \frac{F_{AB} L_{AB}}{E_{AB} A_{AB}} = \frac{(30 \times 10^3 \text{ N}) \cdot (0.3 \text{ m})}{(200 \times 10^6 \text{ Pa}) \cdot (\pi(0.01^2) \text{ m}^2)} = 143 \times 10^{-6} \text{ m} = 0.143 \text{ mm}
\]

\[
\delta_{DE} = \frac{F_{DE} L_{DE}}{E_{DE} A_{DE}} = \frac{(15 \times 10^3 \text{ N}) \cdot (0.3 \text{ m})}{(70 \times 10^9 \text{ Pa}) \cdot (\pi(0.02^2) \text{ m}^2)} = 51 \times 10^{-6} \text{ m} = 0.051 \text{ mm}
\]

\[
\delta_C = \delta_{DE} + (\delta_{AB} - \delta_{DE}) \cdot \left( \frac{0.4 \text{ m}}{0.6 \text{ m}} \right) = 0.113 \text{ mm}
\]

### Statically Indeterminate Problems

In these problems, equations of equilibrium are not enough to solve all the reactions. Hence, equations for compatibility are required.

### Problem 6.

Consider the rod made of an outer layer with material 1 \((E_2 = 90\ \text{ GPa})\) and a core with material 1 \((E_1 = 45\ \text{ GPa})\). It is subjected to \(P = 70\ \text{kN}\). Calculate the stresses developed in each component of the rod.

**Equation of Equilibrium:** The total load \(P\) is carried by both materials. If \(P_1\) is the load carried by material 1 and \(P_2\) is the load carried by material 2

\[
P = P_1 + P_2 = 70\ \text{kN}
\]

**Equation of Compatibility:** Further, the deformations of both materials should be same.

\[
\delta = \delta_1 = \delta_2
\]

\[
\Rightarrow \frac{P_1 L}{E_1 A_1} = \frac{P_2 L}{E_2 A_2}
\]

\[
\Rightarrow P_1 = P_2 \left( \frac{E_1}{E_2} \right) \left( \frac{A_1}{A_2} \right)
\]

\[
\Rightarrow P_1 = P_2 \cdot \left( \frac{90}{45} \right) \cdot \left( \frac{\pi(0.04^2 - 0.02^2)}{\pi(0.02^2)} \right)
\]

\[
\Rightarrow P_1 = P_2 \cdot (2) \cdot (3)
\]

\[
\Rightarrow P_1 = 6P_2
\]
Hence, \( P_1 = 60 \text{ kN}, \) \( P_2 = 10 \text{ kN} \) and

\[
\sigma_1 = \frac{P_1}{A_1} = \frac{60 \times 10^3}{\pi (0.04^2 - 0.02^2)} = 15.91 \text{ MPa}
\]

\[
\sigma_2 = \frac{P_2}{A_2} = \frac{10 \times 10^3}{\pi (0.02^2)} = 7.96 \text{ MPa}
\]

**Problem 7.**

Determine the support reactions in the shown statically indeterminate structure. AC has \( E = 50 \text{ GPa} \) and CD has \( E = 100 \text{ GPa} \).

![Figure 21: Problem 7.](image)

**Equation of Equilibrium:**

\[+ \uparrow \sum F_y = 0\]

\[R_A + R_D = 50 \text{ kN} + 100 \text{ kN} = 150 \text{ kN}\]

**Equation of Compatibility:**

Assume the reaction at D is redundant and \( \delta_L = \) deformation due

Subhayan De, USC
to the load; \( \delta_R = \) deformation due to the reaction. Hence,

\[
\delta = \delta_L + \delta_R = 0
\]

\[
\delta_L = \delta_B + \delta_C + \delta_D
\]

\[
= - \frac{(50 \times 10^3 \text{ N}) \cdot (0.5 \text{ m})}{(50 \times 10^9 \text{ Pa}) \cdot (\pi(0.02^2) \text{ m}^2)} - \frac{(100 \times 10^3 \text{ N}) \cdot (1 \text{ m})}{(50 \times 10^9 \text{ Pa}) \cdot (\pi(0.02^2) \text{ m}^2)} = -1.99 \times 10^{-3} \text{ m}
\]

\[
\delta_R = \frac{(R_D) \cdot (0.5 \text{ m})}{(100 \times 10^9 \text{ Pa}) \cdot (\pi(0.01^2) \text{ m}^2)} + \frac{(R_D) \cdot (1 \text{ m})}{(50 \times 10^9 \text{ Pa}) \cdot (\pi(0.02^2) \text{ m}^2)} = 3.183 \times 10^{-8} R_D
\]

\[
R_D = \frac{1.99 \times 10^{-3}}{3.183 \times 10^{-8}} = 62500 \text{ N} = 62.5 \text{ kN}
\]

\[
R_A = 150 \text{ kN} - R_D = 87.5 \text{ kN}
\]

**Problem 8.**

Solve the same problem as before but allowing a 1 mm gap for the deformation of the bar as shown in the figure.

**Equation of Equilibrium:**

\[
\sum F_y = 0
\]

\[
R_A + R_D = 50 \text{ kN} + 100 \text{ kN} = 150 \text{ kN}
\]

**Equation of Compatibility:** \( \delta_L = \) deformation due to the load; \( \delta_R = \) deformation due to the reaction. Hence,

\[
\delta = \delta_L + \delta_R = -1 \times 10^{-3} \text{ m}
\]

\[
\delta_L = \delta_{AB} + \delta_{BC} + \delta_{CD}
\]

\[
= - \frac{(50 \times 10^3 \text{ N}) \cdot (0.5 \text{ m})}{(50 \times 10^9 \text{ Pa}) \cdot (\pi(0.02^2) \text{ m}^2)} - \frac{(100 \times 10^3 \text{ N}) \cdot (1 \text{ m})}{(50 \times 10^9 \text{ Pa}) \cdot (\pi(0.02^2) \text{ m}^2)} = -1.99 \times 10^{-3} \text{ m}
\]

\[
\delta_R = \frac{(R_D) \cdot (0.5 \text{ m})}{(100 \times 10^9 \text{ Pa}) \cdot (\pi(0.01^2) \text{ m}^2)} + \frac{(R_D) \cdot (1 \text{ m})}{(50 \times 10^9 \text{ Pa}) \cdot (\pi(0.02^2) \text{ m}^2)} = 3.183 \times 10^{-8} R_D
\]

\[
R_D = \frac{1.99 \times 10^{-3} - 1 \times 10^{-3}}{3.183 \times 10^{-8}} = 31250 \text{ N} = 31.25 \text{ kN}
\]

\[
R_A = 150 \text{ kN} - R_D = 118.75 \text{ kN}
\]

**Problem 9.**

Determine the stresses developed in members BE and CF \((E = 70 \text{ GPa}, \text{ radius } = 20 \text{ mm})\). Assume the bar ABCD is rigid.

**Equation of Equilibrium:**

\[
\sum F_x = 0
\]

\[
A_x = 0
\]

\[
\sum M_A = 0
\]

\[
F_{BE} \cdot (0.5 \text{ m}) + F_{CF} \cdot (1 \text{ m}) = (100 \text{ kN}) \cdot (1.5 \text{ m})
\]

\[
F_{BE} + 2F_{CF} = 300 \text{ kN}
\]
Equation of Compatibility:

\[ 2\delta_B = \delta_C \]
\[ \Rightarrow \frac{2F_{BE}L_{BE}}{E_{BE}A_{BE}} = \frac{F_{CF}L_{CF}}{E_{CF}A_{CF}} \]
\[ \Rightarrow \frac{2F_{BE} \cdot (0.5 \text{ m})}{(70 \times 10^9 \text{ Pa}) \cdot (\pi(0.02^2) \text{ m}^2)} = \frac{F_{CF} \cdot (0.5 \text{ m})}{(70 \times 10^9 \text{ Pa}) \cdot (\pi(0.02^2) \text{ m}^2)} \]
\[ \Rightarrow 2F_{BE} = F_{CF} \]

Hence,

\[ F_{BE} = 60 \text{ kN}, \quad F_{CF} = 120 \text{ kN} \]
\[ \sigma_{BE} = \frac{F_{BE}}{A} = \frac{60 \times 10^3 \text{ N}}{\pi(0.02^2) \text{ m}^2} = 47.75 \times 10^6 \text{ Pa} = 47.75 \text{ MPa} \]
\[ \sigma_{CF} = \frac{F_{CF}}{A} = \frac{120 \times 10^3 \text{ N}}{\pi(0.02^2) \text{ m}^2} = 95.5 \times 10^6 \text{ Pa} = 95.5 \text{ MPa} \]