**Problem 4.**

Determine the deformation of a bar under its own weight. What is the equivalent load at the end of the bar that can replace the self-weight?

Consider the deformation of an element of length $dz$. The weight acting on it is

$$P = \rho g A (L - z)$$

where $\rho$ = density of the bar; $A$ = cross-sectional area; $g$ = gravitational acceleration. Hence, the deformation of the element

$$d\delta = \frac{P dz}{EA} = \frac{\rho g A (L - z) dy}{EA}$$

Total deformation of the bar

$$\delta = \int_0^L \frac{\rho g}{E} (L - z) dz$$

$$= \frac{\rho g}{E} \int_0^L (L - z) dz$$

$$= \frac{\rho g}{E} \left( L^2 - L^2/2 \right)$$

$$= \frac{\rho g L^2}{2E}$$

Equivalent force at the end

$$\sigma A$$

$$= (Ee) A$$

$$= \frac{E}{L} \delta A$$

$$= \frac{E \rho g L^2 A}{2EL}$$

$$= \frac{1}{2} \rho g A L = W/2$$

where $W$ is the total weight of the bar.

**Problem 5.**

Determine the deformation at point C.

$$\sum F_y = 0$$

$$\Rightarrow F_{AB} + F_{DE} = 45 \, kN$$

$$\sum M_D = 0$$

$$\Rightarrow - F_{AB} \cdot (0.6 \, m) + (45 \, kN) \cdot (0.4 \, m) = 0$$

$$F_{AB} = 30 \, kN$$

$$F_{DE} = 15 \, kN$$
\[
\delta_{AB} = \frac{F_{AB}L_{AB}}{E_{AB}A_{AB}} = \frac{(30 \times 10^3 \text{ N}) \cdot (0.3 \text{ m})}{(200 \times 10^9 \text{ Pa}) \cdot (\pi(0.01^2) \text{ m}^2)} = 143 \times 10^{-6} \text{ m} = 0.143 \text{ mm}
\]

\[
\delta_{DE} = \frac{F_{DE}L_{DE}}{E_{DE}A_{DE}} = \frac{(15 \times 10^3 \text{ N}) \cdot (0.3 \text{ m})}{(70 \times 10^9 \text{ Pa}) \cdot (\pi(0.02^2) \text{ m}^2)} = 51 \times 10^{-6} \text{ m} = 0.051 \text{ mm}
\]

\[
\delta_C = \delta_{DE} + (\delta_{AB} - \delta_{DE}) \cdot \left( \frac{0.4 \text{ m}}{0.6 \text{ m}} \right) = 0.113 \text{ mm}
\]

**Statically Indeterminate Problems**

In these problems, equations of equilibrium are not enough to solve all the reactions. Hence, equations for compatibility are required.

**Problem 6.**

Consider the rod made of an outer layer with material 1 \((E_2 = 90 \text{ GPa})\) and a core with material 1 \((E_1 = 45 \text{ GPa})\). It is subjected to \(P = 70 \text{ kN}\). Calculate the stresses developed in each component of the rod.

**Equation of Equilibrium:** The total load \(P\) is carried by both materials. If \(P_1\) is the load carried by material 1 and \(P_2\) is the load carried by material 2

\[P = P_1 + P_2 = 70 \text{ kN}\]

**Equation of Compatibility:** Further, the deformations of both materials should be same.

\[\delta = \delta_1 = \delta_2\]

\[\Rightarrow \frac{P_1L}{E_1A_1} = \frac{P_2L}{E_2A_2}\]

\[\Rightarrow P_1 = P_2 \left( \frac{E_1}{E_2} \right) \left( \frac{A_1}{A_2} \right)\]

\[\Rightarrow P_1 = P_2 \cdot \left( \frac{90}{45} \right) \cdot \left( \frac{\pi(0.04^2 - 0.02^2)}{\pi(0.02^2)} \right)\]

\[\Rightarrow P_1 = P_2 \cdot (2) \cdot (3)\]

\[\Rightarrow P_1 = 6P_2\]

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Hence, \( P_1 = 60 \) kN, \( P_2 = 10 \) kN and

\[
\sigma_1 = \frac{P_1}{A_1} = \frac{60 \times 10^3}{\pi (0.04^2 - 0.02^2)} = 15.91 \text{ MPa}
\]

\[
\sigma_2 = \frac{P_2}{A_2} = \frac{10 \times 10^3}{\pi (0.02^2)} = 7.96 \text{ MPa}
\]

**Problem 7.**

Determine the support reactions in the shown statically indeterminate structure. AC has \( E = 50 \) GPa and CD has \( E = 100 \) GPa.

**Figure 21: Problem 7.**

**Equation of Equilibrium:**

\[
\sum F_y = 0 \\
R_A + R_D = 50 \text{ kN} + 100 \text{ kN} = 150 \text{ kN}
\]

**Equation of Compatibility:**

Assume the reaction at D is redundant and \( \delta_L = \) deformation due
to the load; \( \delta_R \) = deformation due to the reaction. Hence,

\[
\delta = \delta_L + \delta_R = 0
\]

\[
\delta_L = \delta_{AB} + \delta_{BC} + \delta_{CD} = -\left( \frac{50 \times 10^3 \text{ N}}{50 \times 10^9 \text{ Pa}} \right) \cdot \frac{(0.02^2 m^2)}{2} - \left( \frac{100 \times 10^3 \text{ N}}{50 \times 10^9 \text{ Pa}} \right) \cdot \frac{(1 m)}{2} = -1.99 \times 10^{-3} \text{ m}
\]

\[
\delta_R = \frac{(R_D) \cdot (0.5 \text{ m})}{50 \times 10^9 \text{ Pa}} \cdot \frac{(0.02^2 m^2)}{2} + \frac{(R_D) \cdot (1 \text{ m})}{50 \times 10^9 \text{ Pa}} \cdot \frac{(0.02^2 m^2)}{2} = 3.183 \times 10^{-8} R_D
\]

\[
\Rightarrow R_D = \frac{1.99 \times 10^{-3} - 1 \times 10^{-3}}{3.183 \times 10^{-8}} = 31250 \text{ N} = 31.25 \text{ kN}
\]

\[
\Rightarrow R_A = 150 \text{ kN} - R_D = 87.5 \text{ kN}
\]

**Problem 9.**

Determine the stresses developed in members BE and CF (\( E = 70 \) GPa, radius = 20 mm).

**Equation of Equilibrium:**

\[
\sum F_x = 0
\]

\[
\Rightarrow A_x = 0
\]

\[
\sum M_A = 0
\]

\[
\Rightarrow F_{BE} \cdot (0.5 \text{ m}) + F_{CF} \cdot (1 \text{ m}) = (100 \text{ kN}) \cdot (1.5 \text{ m})
\]

\[
\Rightarrow F_{BE} + 2F_{CF} = 300 \text{ kN}
\]

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Equation of Compatibility:

\[ 2\delta_B = \delta_C \]
\[ \Rightarrow \frac{2F_{BE}L_{BE}}{E_{BE}A_{BE}} = \frac{F_{CF}L_{CF}}{E_{CF}A_{CF}} \]
\[ \Rightarrow \frac{2F_{BE} \cdot (0.5 \text{ m})}{(70 \times 10^9 \text{ Pa}) \cdot (\pi(0.02^2) \text{ m}^2)} = \frac{F_{CF} \cdot (0.5 \text{ m})}{(70 \times 10^9 \text{ Pa}) \cdot (\pi(0.02^2) \text{ m}^2)} \]
\[ \Rightarrow 2F_{BE} = F_{CF} \]

Hence,

\[ F_{BE} = 60 \text{ kN}, \quad F_{CF} = 120 \text{ kN} \]
\[ \sigma_{BE} = \frac{F_{BE}}{A} = \frac{60 \times 10^3 \text{ N}}{\pi(0.02^2) \text{ m}^2} = 47.75 \times 10^6 \text{ Pa} = 47.75 \text{ MPa} \]
\[ \sigma_{CF} = \frac{F_{CF}}{A} = \frac{120 \times 10^3 \text{ N}}{\pi(0.02^2) \text{ m}^2} = 95.5 \times 10^6 \text{ Pa} = 95.5 \text{ MPa} \]

Figure 23: Problem 9.