Example 1.

Seven fair coins are flipped. What is the probability that the outcomes are two heads and five tails?

Denote the random variable $X$ as the number of heads (successes) obtained.

Hence, $X$ is binomial with $n = 7$ and $p = 1/2$.

So,

$$P(X = 2) = \binom{7}{2}(1/2)^2(1 - 1/2)^5 \approx 0.1641$$

Example 2.

An aircraft engine fails with probability $1 - p$ during a flight independent of other engines.

The plane can fly if at least half of its engines are running.

What can you say about $p$ if the engineer says two-engine plane is safer than a four-engine one?

Let us denote the number of engines running during a flight for a four-engine plane by $X_4$ and for a two-engine plane by $X_2$. Note that $X_4$ is binomial with parameters $n = 4$ and $p$ and $X_2$ is binomial with parameters $n = 2$ and $p$.

Hence, the probability that a four-engine plane will complete its flight

$$P(X_4 \geq 2) = P(X_4 = 2) + P(X_4 = 3) + P(X_4 = 4)$$

$$= \binom{4}{2}p^2(1-p)^2 + \binom{4}{3}p^3(1-p)^1 + \binom{4}{4}p^4(1-p)^0$$

$$= 6p^2(1-p)^2 + 4p^3(1-p) + p^4$$

Similarly, the probability that a two-engine plane will complete its flight

$$P(X_2 \geq 1) = P(X_4 = 1) + P(X_4 = 2)$$

$$= \binom{2}{1}p(1-p) + \binom{2}{2}p^2(1-p)^0$$

$$= 2p(1-p) + p^2$$

Hence, the two-engine plane is safer than a four-engine one if

$$2p(1-p) + p^2 \geq 6p^2(1-p)^2 + 4p^3(1-p) + p^4$$

$$3p^2 - 8p^2 + 7p - 2 \leq 0$$

$$(p - 1)^2(3p - 2) \leq 0$$

$$3p - 2 \leq 0 \quad [\text{since } p \neq 1]$$

$$p \leq 2/3$$

Example 3.

The probability that a traffic signal will malfunction is 0.01. Calculate the probability that in a city with 100 traffic signals five or more will malfunction.

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The random variable $X$ denotes the number of malfunctioning traffic signals. Hence, $X$ is binomial with parameters $n = 100$ and $p = 0.01$ (i.e., $n$ large, $p$ small).

Using the Poisson approximation of binomial, $X$ is approximately Poisson distributed with parameter $\lambda = np = 1$.

Hence,

$$P(X \geq 5) = 1 - P(X < 5) \approx 1 - \sum_{i=0}^{4} \frac{\lambda^i}{i!} e^{-\lambda}$$

$$= 1 - e^{-1} \left[ 1 + \frac{1}{1!} + \frac{1^2}{2!} + \frac{1^3}{3!} + \frac{1^4}{4!} \right]$$

$$= 0.0037$$

**Example 4.**

In a fast-food joint, during rush-hour customer arrives at a rate of $\alpha$ per minute. It is given that the arrival of the customer during a time period is Poisson distributed. Find the probabilities that there are no customers and more than 10 customers in $T$ minutes during rush-hour.

Denote the number of customers by $X$ in $T$ minutes during rush-hour.

Hence, $X$ is Poisson distributed with parameter $\lambda = \alpha T$.

So, the probability that there are no customers in $T$ minutes during rush-hour

$$P(X = 0) = \frac{(\alpha T)^0}{0!} e^{-\alpha T} = e^{-\alpha T}$$

The probability that there are more than 10 customers in $T$ minutes during rush-hour

$$P(X \geq 10) = 1 - P(X < 10) = 1 - \sum_{i=0}^{10} \frac{(\alpha T)^i}{i!} e^{-\alpha T}$$

**Example 5.**

$X_i, i = 1, \ldots, 10$ are independent Poisson random variables with mean 1.

Get a bound on $P \left( \sum_{i=1}^{10} X_i \geq 15 \right)$.

Using Markov inequality,

$$P \left( \sum_{i=1}^{10} X_i \geq 15 \right) \leq \frac{\mathbb{E}[\sum_{i=1}^{10} X_i]}{15}$$

$$= \sum_{i=1}^{10} \frac{\mathbb{E}[X_i]}{15}$$

$$= \frac{10 \cdot 1}{15} = \frac{2}{3}$$

**Example 6.**

$X$ and $Y$ are independent binomial random variables with parameters $(n, p)$ and $(m, p)$, respectively. Show that $X + Y$ is also binomial with $(n + m, p)$.

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\[ P(X + Y = k) = \sum_{i=0}^{k} P(X = i, Y = k - i) \]
\[ = \sum_{i=0}^{k} P(X = i)P(Y = k - i) \quad \text{[by independence]} \]
\[ = \sum_{i=0}^{k} \binom{n}{i} p^i (1-p)^{n-i} \binom{m}{k-i} p^{k-i} (1-p)^{m-k+i} \]
\[ = p^k (1-p)^{n+m-k} \sum_{i=0}^{k} \binom{n}{i} \binom{m}{k-i} \]
\[ = \binom{n+m}{k} p^k (1-p)^{n+m-k} \]

**Example 7.**

\( X \) and \( Y \) are independent Poisson random variables with parameters \( \lambda_1 \) and \( \lambda_2 \), respectively. Show that \( X + Y \) is also Poisson with mean \( \lambda_1 + \lambda_2 \).

\[ P(X + Y = k) = \sum_{i=0}^{k} P(X = i, Y = k - i) \]
\[ = \sum_{i=0}^{k} P(X = i)P(Y = k - i) \quad \text{[by independence]} \]
\[ = \sum_{i=0}^{k} e^{-\lambda_1} \frac{\lambda_1^i}{i!} e^{-\lambda_2} \frac{\lambda_2^{k-i}}{(k-i)!} \]
\[ = e^{-(\lambda_1 + \lambda_2)} \sum_{i=0}^{k} \frac{\lambda_1^i \lambda_2^{k-i}}{i!(k-i)!} \]
\[ = \frac{e^{-(\lambda_1 + \lambda_2)}}{k!} \sum_{i=0}^{k} \frac{k!}{i!(k-i)!} \lambda_1^i \lambda_2^{k-i} \]
\[ = \frac{e^{-(\lambda_1 + \lambda_2)}}{k!} (\lambda_1 + \lambda_2)^k \]

**Example 8.**

\( X \) and \( Y \) are independent exponential random variables with parameters \( \lambda \) and \((m, p)\), respectively. Estimate the probability density of \( Z = X + Y \).

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\[ F_Z(z) = \int_{-\infty}^{\infty} \int_{-\infty}^{z-y} f_{X,Y}(x,y) dx \, dy \]
\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{z-y} f_X(x) f_Y(y) dy \, dx \]
\[ = \int_{-\infty}^{\infty} F_X(z-y) f_Y(y) dy \]

\[ f_Z(z) = \frac{d}{dz} F_Z(z) = \frac{d}{dz} \int_{-\infty}^{z} F_X(z-y) f_Y(y) dy \]
\[ = \int_{-\infty}^{\infty} \frac{d}{dz} F_X(z-y) f_Y(y) dy \]
\[ = \int_{-\infty}^{\infty} f_X(z-y) f_Y(y) dy \]

Hence,

\[ f_Z(z) = \int_{0}^{z} \lambda^2 e^{-\lambda(z-y)} e^{-\lambda y} dy \quad 0 < y < z \]
\[ = \int_{0}^{z} \lambda^2 e^{-\lambda z} dy \]
\[ = \lambda^2 z e^{-\lambda z} \]
\[ = \frac{(\lambda z)^{2-1}}{\Gamma(2)} \lambda e^{-\lambda z} \]

So, \( X + Y \sim \text{Gamma}(\lambda,2) \).

**Example 9.**

\( X \) and \( Y \) are independent uniform random variables on \((0, 1)\). Estimate the probability density of \( Z = X + Y \).

The probability density of \( X \) and \( Y \) are

\[ f_X(x) = f_Y(y) = \begin{cases} 1, & 0 < x, y < 1 \\ 0, & \text{otherwise} \end{cases} \]

Hence,

\[ f_Z(z) = \int_{0}^{1} f_X(z-y) f_Y(y) dy = \int_{0}^{1} f_X(z-y) dy \]

For \( 0 \leq z \leq 1 \),

\[ f_Z(z) = \int_{0}^{z} dy = z \]

For \( 1 < z < 2 \),

\[ f_Z(z) = \int_{z-1}^{1} dy = 2 - z \]

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This gives a triangular density

\[ f_Z(z) = \begin{cases} 
  z, & 0 \leq z \leq 1 \\
  2 - z, & 1 < z < 2 \\
  0, & \text{otherwise}
\end{cases} \]

**Example 10.**

Order statistics \( X_1, X_2, \ldots, X_n \) are independent and identically distributed with CDF \( F(x) \) and pdf \( f(x) \). If \( X_{(i)} \) is the \( i \)th smallest RV then determine the pdf of \( X_{(i)} \).

\[ F_{X_{(i)}}(x) = P(X_{(i)} \leq x) = \sum_{k=i}^{n} [F(x)]^k [1 - F(x)]^{n-k} \]

\[ \Rightarrow \quad f_{X_{(i)}}(x) = \frac{d}{dx} F_{X_{(i)}}(x) = f(x) \sum_{k=i}^{n} \binom{n}{k} k[F(x)]^{k-1} [1 - F(x)]^{n-k} - f(x) \sum_{k=i}^{n} \binom{n}{k} (n - k)[F(x)]^n [1 - F(x)]^{n-k-1} \]

\[ = f(x) \sum_{k=i}^{n} \frac{n!}{(n-k)!k!} k[F(x)]^{k-1} [1 - F(x)]^{n-k} - f(x) \sum_{k=i}^{n} \frac{n!}{(n-k)!k!} (n - k)[F(x)]^k [1 - F(x)]^{n-k-1} \]

\[ = f(x) \sum_{k=i}^{n} \frac{n!}{(n-k)!(k-1)!} [F(x)]^{k-1} [1 - F(x)]^{n-k} - f(x) \sum_{k=i}^{n} \frac{n!}{(n-k-1)!k!} [F(x)]^k [1 - F(x)]^{n-k-1} \]

\[ = f(x) \sum_{k=i}^{n} \frac{n!}{(n-k)!(k-1)!} [F(x)]^{k-1} [1 - F(x)]^{n-k} - f(x) \sum_{j=i+1}^{n} \frac{n!}{(n-j)!(j-1)!} [F(x)]^{j-1} [1 - F(x)]^{n-j} \]

\[ = \frac{n!}{(n-i)!(i-1)!} f(x) [F(x)]^{i-1} [1 - F(x)]^{n-i} \]
Example 11.

If $Z_1, Z_2, \ldots, Z_n$ are standard Gaussian random variables (i.e., with zero mean and standard deviation 1) and the random variables $X_1, X_2, \ldots, X_m$ are given by

\[
X_1 = a_{11}Z_1 + \cdots + a_{1n}Z_n + \mu_1 \\
X_2 = a_{21}Z_1 + \cdots + a_{2n}Z_n + \mu_2 \\
\vdots \\
X_i = a_{i1}Z_1 + \cdots + a_{in}Z_n + \mu_i \\
\vdots \\
X_m = a_{m1}Z_1 + \cdots + a_{mn}Z_n + \mu_m
\]

i.e., $X = AZ + \mu$ where $X = [X_1, X_2, \ldots, X_m]^T$, $A = [a_{ij}]$, $Z = [Z_1, Z_2, \ldots, Z_n]^T$, and $\mu = [\mu_1, \mu_2, \ldots, \mu_m]$.

Hence,

\[
E[X_i] = \mu_i \\
\text{Var}(X_i) = \sum_{j=1}^{n} a_{ij}^2
\]

\[
E[X] = \mu \\
\text{Cov}(X) = AA^T
\]

In general, when $Y = AX$ with $\text{Cov}(X) = \Sigma$

\[
\text{Cov}(Y) = \text{Cov}(AX) = E[(AX - E[AX])(AX - E[AX])^T] \\
= E[(AX - AE[X])(AX - AE[X])^T] \\
= E[A(X - E[X])(X - E[X])^T A^T] \\
= A E[(X - E[X])(X - E[X])^T] A^T \\
= A \Sigma A^T
\]