Example 9.

Let $X$ be a random variable with probability density

$$f(x) = \begin{cases} 
  c(1 - x^2), & -1 < x < 1 \\
  0, & \text{otherwise}
\end{cases}$$

Determine the value of $c$ and $F(x)$.

$$F(\infty) = \int_{-\infty}^{\infty} f(x) dx = 1$$

$$\Rightarrow \int_{-1}^{1} c(1 - x^2) dx = 1$$

$$\Rightarrow c \left[ x - \frac{x^3}{3} \right]_{-1}^{1} = 1$$

$$\Rightarrow c \left[ (1 - 1/3) - (-1 + 1/3) \right] = 1$$

$$\Rightarrow c \left[ 2/3 + 2/3 \right] = 1$$

$$\Rightarrow c = 3/4$$

The cumulative distribution function

$$F(x) = \int_{-\infty}^{x} f(a) da$$

$$= \int_{-1}^{x} c(1 - a^2) da$$

$$= c \left[ a - \frac{a^3}{3} \right]_{-1}^{x}$$

$$= \frac{3}{4} \left[ x - \frac{x^3}{3} + \frac{2}{3} \right]$$

Example 10.

Two teams are playing a series of games. The first team to win $i$ games is the winner.

Team A wins a game with probability $p$.

Team B wins with probability $1 - p$.

If $i = 4$, what is the probability that a total of 7 games are played?

$$P\{7 \text{ games are played}\} = \binom{6}{3} p^3 (1-p)^3$$

[7 games are played when first 6 are 3 wins and 3 losses for team A or B]

This probability is maximized when

$$\frac{d}{dp} \left\{ \binom{6}{3} p^3 (1-p)^3 \right\} = 0$$

$$\Rightarrow 20[3p^2(1-p)^3 - p^3 \cdot 3(1-p)^2] = 0$$

$$\Rightarrow 60p^2(1-p)^2(1-2p) = 0$$

$$\Rightarrow p = 1/2$$

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Expected number of games played when $i = 3$

Let $X$ denote the total no. of games played.

\[
P(X = 3) = P(\text{A wins all 3 games} \cup \text{B wins all 3 games})
= P(\text{A wins all 3 games}) + P(\text{B wins all 3 games})
= p^3 + (1 - p)^3
\]

\[
P(X = 4) = P(X = 4 \text{ and A wins the series } 3-1) + P(X = 4 \text{ and B wins the series } 3-1)
= P(X = 4 \text{ and A has 2 wins in the first 3 games})
+ P(X = 4 \text{ and B has 2 wins in the first 3 games})
= \left(\frac{3}{2}\right)p^2(1 - p)p + \left(\frac{3}{2}\right)(1 - p)^2p(1 - p)
= 3p^3(1 - p) + 3p(1 - p)^3
= 3p(1 - p)[p^2 + (1 - p)^2]
\]

\[
P(X = 5) = P(X = 5 \text{ and A wins the series}) + P(X = 5 \text{ and B wins the series})
= P(X = 5 \text{ and A wins 2 out of first 4 games and the last one})
+ P(X = 5 \text{ and B wins 2 out of first 4 games and the last one})
= \left(\frac{4}{2}\right)p^2(1 - p)^2p + \left(\frac{4}{2}\right)(1 - p)^2p^2(1 - p)
= 6p^2(1 - p)^2p + 6(1 - p)^2p^2(1 - p)
= 6p^2(1 - p)^2[p + 1 - p]
= 6p^2(1 - p)^2
\]

Hence,

\[
\mathbb{E}[X] = 3 \cdot P(X = 3) + 4 \cdot P(X = 4) + 5 \cdot P(X = 5)
= 3[p^3 + (1 - p)^3] + 12p(1 - p)[p^2 + (1 - p)^2] + 30p^2(1 - p)^2
\]

This is maximized when $p = 1/2$.

**Example 11.**

The useful life $T$ (in hours) of light bulbs is described by the following density function

\[
f(t) = \begin{cases} 
\lambda e^{-\lambda t} & t \geq 0 \\
0 & t < 0
\end{cases}
\]

where $\lambda$ is a constant.
The cumulative distribution function

\[
F(t) = \int_{-\infty}^{t} f(\tau) d\tau
= \int_{0}^{t} \lambda e^{-\lambda \tau} d\tau
= \lambda \left[ -\frac{1}{\lambda} e^{-\lambda \tau} \right]_{0}^{t}
= [-e^{-\lambda t} - (-1)]
= 1 - e^{-\lambda t}
\]

Figure 3.4: pdf and CDF of \(T\).

The mean life of the light bulb is

\[
\mathbb{E}[T] = \int_{-\infty}^{\infty} tf(t) dt
= \int_{0}^{\infty} t \lambda e^{-\lambda t} dt
= -\lambda e^{-\lambda t}\bigg|_{0}^{\infty} + \int_{0}^{\infty} e^{-\lambda t} dt
= 0 - \lambda \left[ e^{-\lambda t} \right]_{0}^{\infty}
= \frac{1}{\lambda}
\]

Median: \(F(t_m) = \int_{0}^{t_m} \lambda e^{-\lambda t} dt = 0.5\). This gives \(t_m = \frac{1}{\lambda} [\log(0.5)] = 0.693/\lambda = 0.693\mathbb{E}[T]\).

\[
\text{Var}(T) = \int_{0}^{\infty} (t - 1/\lambda)^2 \lambda e^{-\lambda t} dt = 1/\lambda^2
\]

**Example 12.**

\(X\) and \(Y\) are independent random variables with means \(\mu_X\) and \(\mu_Y\), respectively, and variances \(\sigma_X^2\) and \(\sigma_Y^2\), respectively.

\(\mathbb{E}[XY] = ?\) and \(\text{Var}(XY) = ?\)

\[
\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y] = \mu_X \mu_Y
\]

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\[
\mathbb{E}[(XY)^2] = \mathbb{E}[X^2] \mathbb{E}[Y^2] \\
= \{\text{Var}(X) + (\mathbb{E}[X])^2\} \{\text{Var}(Y) + (\mathbb{E}[Y])^2\} \\
= (\sigma_X^2 + \mu_X^2)(\sigma_Y^2 + \mu_Y^2)
\]

Hence,
\[
\text{Var}(XY) = \mathbb{E}[(XY)^2] - (\mathbb{E}[XY])^2 \\
= (\sigma_X^2 + \mu_X^2)(\sigma_Y^2 + \mu_Y^2) - \mu_X^2 \mu_Y^2 \\
= \sigma_X^2 \sigma_Y^2 + \sigma_X^2 \mu_Y^2 + \sigma_Y^2 \mu_X^2
\]

**Example 13.**

At the graduation ceremony \(N\) students throw their hats and the select one at random. What is the expected number of students who will get their own hats back?

Let \(X\) denote the no. of students who select their own hat

\[
X_i = \begin{cases} 
1, & \text{if } i\text{th student selects own hat} \\
0, & \text{otherwise}
\end{cases}
\]

Hence, \(X = \sum_{i=1}^{N} X_i\)

Also, \(P(X_i = 1) = \text{probability that } i\text{th student select own hat} = \frac{1}{N}\)

\[
\mathbb{E}[X] = \mathbb{E} \left[ \sum_{i=1}^{N} X_i \right] \\
= \sum_{i=1}^{N} \mathbb{E}[X_i] \\
= \sum_{i=1}^{N} \frac{1}{N} \\
= N \cdot \frac{1}{N} \\
= 1
\]

**Example 14.**

A basket has \(n\) Red balls and \(m\) Blue balls. \(k\) balls are selected at random from the basket.

Let \(X\) denote the number of Red balls selected.

\(P(X = i) = ?\) and \(\mathbb{E}[X] = ?\)

\[
P(X = i) = \binom{n}{i} \binom{m}{k-i} \binom{m+n}{k}
\]

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Let us denote

\[ X_j = \begin{cases} 
  1, & \text{if } j \text{th ball selected is Red} \\
  0, & \text{otherwise } j = 1, 2, \ldots, k
\end{cases} \]

Hence, \( X = \sum_{j=1}^{k} X_j \)

\[ \mathbb{E}[X] = \mathbb{E} \left[ \sum_{j=1}^{k} X_j \right] = \sum_{j=1}^{k} \mathbb{E}[X_j] \]

Now,

\[ \mathbb{E}[X_j] = 1 \cdot P(X_j = 1) + 0 \cdot P(X_j = 0) = 1 \cdot \frac{n}{n+m} + 0 = \frac{n}{n+m} \]

This gives

\[ \mathbb{E}[X] = \sum_{j=1}^{k} \mathbb{E}[X_j] = \sum_{j=1}^{k} \frac{n}{n+m} = \frac{nk}{n+m} \]

OR

Define

\[ Y_j = \begin{cases} 
  1, & \text{if Red ball } j \text{ is selected} \\
  0, & \text{otherwise } j = 1, 2, \ldots, n
\end{cases} \]

Again, \( X = \sum_{j=1}^{n} Y_j \)

\[ \mathbb{E}[X] = \sum_{j=1}^{n} \mathbb{E}[Y_j] \]

Now,

\[ \mathbb{E}[Y_j] = 1 \cdot P(Y_j = 1) + 0 \cdot P(Y_j = 0) = P(Y_j = 1) = \frac{k}{n+m} \]

This gives

\[ \mathbb{E}[X] = \sum_{j=1}^{n} \mathbb{E}[Y_j] = \sum_{j=1}^{n} \frac{k}{n+m} = \frac{nk}{n+m} \]

Example 15.

\( K(t) = \log \left( \mathbb{E}[e^{tX}] \right) \). Show that \( K'(0) = \mathbb{E}[X], K''(0) = \text{Var}(X) \).
$K(t) = \log(\mathbb{E}[e^{tX}])$

$K'(t) = \frac{d}{dt} K(t)$

$= \frac{\mathbb{E}[X e^{tX}]}{\mathbb{E}[e^{tX}]}$

$K''(t) = \frac{d^2}{dt^2} K(t)$

$= \frac{\mathbb{E}[e^{tX}]\mathbb{E}[X^2 e^{tX}] - (\mathbb{E}[e^{tX}])^2}{(\mathbb{E}[e^{tX}])^2}$

Hence, $K'(0) = \mathbb{E}[X]$, $K''(0) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \text{Var}(X)$. 

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