Elements of Probability

Probability - two interpretations

Frequency interpretation
Subjective

Sample space: The set of all possible outcomes of an experiment (denoted by 'S' or 'Ω')

Any subset E of the sample space is known as an event.

Ex. A coin is to be tossed until a head appears twice in a row.

Sample space, $S = \{(H, H), (T, H, H), (H, T, H, H), (T, T, H, H), \ldots\}$

We can also write this in a different way

$S = \{(e_1, e_2, \ldots, e_{n-1}, e_n), n \geq 2\}$

where $e_i$ is either $H$ or $T$

and $e_{n-1} = e_n = H$
Axiomatic definition of Probability

Sample space: \( S \)
Event: \( E \)

Probability of event \( E \), \( P(E) \) satisfies:

Axiom 1: \( 0 \leq P(E) \leq 1 \)

Axiom 2: \( P(S) = 1 \)

Axiom 3: Mutually exclusive events
\( E_1, E_2, \ldots \) \( (i.e., E_i \cap E_j = \emptyset \) when \( i \neq j \) \)

\[
P(\bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} P(E_i), \quad n = 1, 2, \ldots, \infty
\]

Corollary: (i) \( E \) and \( E^c \) are mutually exclusive
\( E \cup E^c = S \)
\( P(E \cup E^c) = P(S) = 1 \)
\( P(E^c) = 1 - P(E) \)

(ii) Two events \( E \) and \( F \)
\( P(E \cup F) = P(E) + P(F) - P(E \cap F) \)

Note: \( E \cap F \) can also be written as \( EF \)
\[ P(E_1 \cup E_2 \cup E_3 \cup \ldots \cup E_n) = \sum_{i} P(E_i) - \sum_{i,j} P(E_i \cap E_j) + \sum_{i,j,k} P(E_i \cap E_j \cap E_k) \]

\[ - \sum_{i,j,k,l} P(E_i \cap E_j \cap E_k \cap E_l) + \ldots \]

\[ + (-1)^{n+1} P(E_1 \cap E_2 \cap \ldots \cap E_n) \]

**Inclusion-Exclusion Identity**

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**Conditional Probabilities**

The probability that \( E \) occurs given that \( F \) has occurred.

\[ P(E|F) = \frac{P(EF)}{P(F)} \]

**Independent Events**

If \( P(EF) = P(E)P(F) \), then \( E \) and \( F \) are independent.

\[ P(E|F) = P(E) \]

We also have \[ P(E|F) = P(E) \]
Ex. A coin is to be tossed until a head appears twice in a row.
What is the probability that it will be tossed exactly four times?

\[ P \{ 4 \text{ tosses} \} = P \{ (T, T, H, H) \text{ and } (H, T, H, H) \} \]

\[ = P \{ (T, T, H, H) \} + P \{ (H, T, H, H) \} \]

\[ = \left( \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \right) + \left( \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \right) \]

\[ = \frac{1}{8} \]

Ex. E, F, G three events.

(a) only F occurs

\[ F \cap \overline{E} \cap \overline{G} \]

(b) at least two events occur

\[ (E \cap F) \cup (E \cap G) \cup (F \cap G) \]

(c) at least one event occurs

\[ E \cup F \cup G \]

(d) all three events occur

\[ E \cap F \cap G \]
(e) at most two occur

\[(ENFNG)^c\]

(f) none occurs \[(EUFUG)^c = E^c F^c G^c\]

Ex. Boole's inequality:

\[P(\bigcup_{i=1}^{n} E_i) \leq \sum_{i=1}^{n} P(E_i)\]

Proof:

\[\bigcup_{i=1}^{n} E_i = E_1 \cup E_1^c \cup E_2 \cup E_2^c \cup E_3 \cup \ldots \]

\[\ldots \cup E_i^c \cup E_{i-1}^c \cup E_n\]

Denote \(F_i = E_i\)

\[F_2 = E_1^c \cap E_2\]

\[F_n = E_1^c \ldots E_{n-1}^c \cup E_n\]

\[\bigcup_{i=1}^{n} E_i = \bigcup_{i=1}^{n} F_i\]

But \(F_i\) are mutually exclusive.

\[\Rightarrow P(\bigcup_{i=1}^{n} E_i) = P(\bigcup_{i=1}^{n} F_i) = \sum_{i=1}^{n} P(F_i) \leq \sum_{i=1}^{n} P(E_i)\]
Remember, \( P(F_i) = P(E_i \cap E_i^{c}) \)

\[ \leq P(E_i) \]

**Ex.** A deck of 52 playing cards, containing all 4 aces, is randomly divided into 4 piles of 13 cards each.

(a) \( P(E_1) = P(\text{the first pile has exactly 1 ace}) \)

\[ \ast = \binom{4}{1} \times \binom{48}{12} = \frac{4c_1 \times 48c_{12}}{52c_{13}} \]

\[ = 0.4388 \]

(b) Similarly, \( P(E_2) = P(\text{the 2nd pile has exactly 1 ace}) \)

\[ = 0.4388 \]

\[ P(E_4) = P(E_3) = P(E_2) = P(E_1) \]

Now, \( P(E_2 | E_1) = \frac{P(E_2 \cap E_1)}{P(E_1)} = \frac{\binom{3}{1} \times \binom{36}{12}}{\binom{39}{13}} \approx 0.4623 \)

\[ P(E_3 | E_1 E_2) = \frac{\binom{2}{1} \times \binom{24}{12}}{\binom{26}{13}} = 0.52 \]
\[ P(E_4 | E_1, E_2, E_3) = 1 \]

\[ \Rightarrow P(E_1, E_2, E_3, E_4) = P(E_1) P(E_2 | E_1) P(E_3 | E_1, E_2) P(E_4 | E_1, E_2, E_3) \]

\[ = 0.1055 \]

Ex. N graduating students throw their graduate cap and then, randomly takes one.

The probability that none of the N students gets his/her own cap is

\[ P(\text{no one selects own cap}) \]

\[ = 1 - P(E_1 U E_2 U \ldots U E_N) \]

( where \( E_i \) = event where \( i^{th} \) student gets his/her own cap)

\[ = 1 - \left[ \sum_{i=1}^{N} P(E_{i_1}) - \sum_{i_1 < i_2} P(E_{i_1}, E_{i_2}) + \cdots \right. \]

\[ + \left. (-1)^{N+1} P(E_1, E_2, \ldots, E_N) \right] \]

\[ = 1 - \sum_{i_1} P(E_{i_1}) + \sum_{i_1 < i_2} P(E_{i_1}, E_{i_2}) - \cdots \]

\[ - (-1)^{N+1} P(E_1, E_2, \ldots, E_N) \]
Now, \( P(E_{i_1}, E_{i_2}, \ldots, E_{i_k}) \)
\[= \frac{(N-k)!}{N!} \]

\[= \sum_{i_1 < \ldots < i_k} P(E_{i_1}, E_{i_2}, \ldots, E_{i_k}) = \sum_{i_1 < \ldots < i_k} \frac{(N-k)!}{N!} \]
\[= \binom{N}{k} \frac{(N-k)!}{N!} \]
\[= \frac{N!}{k!(N-k)!} \cdot \frac{(N-k)!}{N!} \]
\[= \frac{1}{k!} \]

\[\Rightarrow P(\text{no one selects own cap}) \]
\[= 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots - (-1)^{N+1} \frac{1}{N!} \]
\[= 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^N \frac{1}{N!} \]
Ex. **Bridge Lifetimes**

The probability that in any year a flood takes place = \( p \)

The probability that the bridge will become obsolete in year \( i \) given that it has not become obsolete prior to year \( i \) = \( r_i \)

Define events:

\( A_i \) = a critical flood destroys the bridge in year \( i \)

\( B_i \) = the bridge becomes obsolete in year \( i \)

\( \therefore P(A_i) = p \) and \( r_i \) = \( P(B_i \mid B_{i-1}) \)

The probability that the bridge's life does not end in the first year

\[
= P(A_1^c \cap B_1^c) = P(A_1^c) P(B_1^c) \\
= (1 - P(A_1)) [1 - P(B_1)] \\
= (1 - p)(1 - r_1)
\]

For 2\(^{nd}\) year,

\[
P[(A_1^c \cap B_1^c) \cap (A_2^c \cap B_2^c)] \\
= P(A_1^c \cap B_1^c) P(A_2^c \cap B_2^c \mid A_1^c \cap B_1^c) \\
= P(A_1^c \cap B_1^c) P(A_2^c \mid B_2^c \cap (A_1^c \cap B_1^c)) P(B_2^c \mid A_1^c \cap B_1^c)
\]
As the events $A_i$ and $B_i$ are independent,

$$P(A_2^c \mid B_2^c \cap A_1^c \cap B_1^c) = P(A_2^c) = 1 - P$$

$$P(B_2^c \mid A_1^c \cap B_1^c) = P(B_2^c \mid B_1^c) = 1 - r_2$$

Hence,

$$P[(A_1^c \cap B_1^c) \cap (A_2^c \cap B_2^c)]$$

$$= (1 - p)(1 - r_1)(1 - p)(1 - r_2)$$

$$= (1 - p)^2 (1 - r_1)(1 - r_2)$$

For the $n$th year,

$$P[\text{survival through } n \text{ years}]$$

$$= P(A_1^c \cap B_1^c \cap \ldots \cap A_n^c \cap B_n^c)$$

$$= (1 - p)^n \prod_{i=1}^{n} (1 - r_i)$$

To calculate the probability that the life of the bridge ends in year $n$:

It means the bridge has survived $(n-1)$ years and

$$P[\text{survival through } (n-1) \text{ years}]$$

$$= (1 - p)^{n-1} \prod_{i=1}^{n-1} (1 - r_i)$$
Now, \( p \) (the bridge's life ends in year \( n \))

\[
= P(A_n \cup B_n) \text{ survival through } n-1 \text{ years) } \\
= P(A_n) \text{ previous survival) } + P(B_n | \text{ previous survival) } \\
- P(A_n \cap B_n | \text{ previous survival) } \\
= P(A_n) + P(B_n) - P(A_n) P(B_n | B_{n-1}^c) \\
= \rho + r_n - \rho r_n
\]

Hence, \( p \) (life ends in year \( n \))

\[
= (\rho + r_n - \rho r_n) (1-\rho)^{n-1} \prod_{i=1}^{n-1} (1-r_i)
\]
Total Probability

- An event $A$
- $N$ mutually exclusive events $B_n$, $n=1, 2, \ldots, N$

where $\bigcup_{n=1}^{N} B_n = S$

Then $P(A) = \sum_{n=1}^{N} P(A \mid B_n) \cdot P(B_n)$

Proof: $A = A \cap S = A \cap \bigcup_{n=1}^{N} B_n = \bigcup_{n=1}^{N} (A \cap B_n)$

$(A \cap B_n)$ are mutually exclusive events.

$\therefore P(A) = P(A \cap S) = P\left[ \bigcup_{n=1}^{N} (A \cap B_n) \right]$

$= \sum_{n=1}^{N} P(A \cap B_n)$

$= \sum_{n=1}^{N} P(A \mid B_n) \cdot P(B_n)$

Bayes' Theorem

$P(B_n \mid A) = \frac{P(B_n \cap A)}{P(A)}$

Proof: $P(A \mid B_n) = \frac{P(A \cap B_n)}{P(B_n)} = \frac{P(B_n \mid A) \cdot P(A)}{P(B_n)}$
\[
\therefore P(B_n | A) = \frac{P(A | B_n) P(B_n)}{P(A)} = \frac{P(B_n \cap A)}{P(A)}
\]

**Ex:** Urn 1: 5 red balls & 7 blue balls
Urn 2: 3 " " 6 & 12 " "

Flip a fair coin. If the outcome is heads, then select a ball from urn 1.
If the outcome is tails, then select a ball from urn 2.

A red ball is selected.
What is the probability that the coin landed tails?

\( R \) = event where a red ball is selected

\( P(T | R) = ? \)

Now, \( P(R | T) = \frac{3}{15} = \frac{1}{5} \) [\( \because \) 2nd urn is selected]

\[ P(T) = P(H) = \frac{1}{2} \]

\[ P(R | H) = \frac{5}{12} \] [\( \because \) 1st urn is selected]
From total probability,

\[ P(R) = P(R|T)P(T) + P(R|H)P(H) \]

\[ = \frac{1}{5} \times \frac{1}{2} + \frac{5}{12} \times \frac{1}{2} \]

\[ = \frac{37}{120} \]

Using Bayes' Theorem,

\[ P(T|R) = \frac{P(R|T)P(T)}{P(R)} \]

\[ = \frac{\frac{1}{5} \times \frac{1}{2}}{\frac{37}{120}} \]

\[ = \frac{12}{37} \]
Ex. **Quality of concrete**

80% of concrete mixes are good. On site tests are performed but the tests are only 90% reliable.

Define the events:

- $G =$ good concrete mix
- $T =$ the concrete mix passes the test

We know,

\[ P(G) = 0.8 \]
\[ P(T|G) = 0.9 \]
\[ P(T|G^c) = 0.1 \]

What is the probability that the concrete mix is good if it passes the test?

\[ P(G|T) = ? \]

Using Bayes' theorem,

\[ P(G|T) = \frac{P(T|G)P(G)}{P(T|G)P(G) + P(T|G^c)P(G^c)} \]

\[ = \frac{0.9 \times 0.8}{0.9 \times 0.8 + 0.1 \times 0.2} = 0.9730 \]