Concentration of measure for the number of isolated vertices in the Erdős-Rényi random graph by size bias couplings

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Abstract

Let $Y$ be a nonnegative random variable with mean $\mu$ and let $Y^*$, defined on the same space as $Y$, have the $Y$-size biased distribution, that is, the distribution characterized by

$$E[Yf(Y)] = \mu E[f(Y^*)]$$

for all functions $f$ for which these expectations exist.

The size bias coupling of $Y$ to $Y^*$ can be used to obtain the following concentration of measure result when $Y$ counts the number of isolated vertices in an Erdős-Rényi random graph model on $n$ edges with edge probability $p$. With $\sigma^2$ denoting the variance of $Y$,

$$P\left(\frac{Y - \mu}{\sigma} \geq t\right) \leq \inf_{\theta \geq 0} \exp(-\theta t + H(\theta)) \quad \text{where} \quad H(\theta) = \frac{\mu}{2\sigma^2} \int_0^{\theta} s\gamma_s ds$$

with

$$\gamma_s = 2e^{2s} \left(1 + \frac{pe^s}{1 - p}\right)^n + (1 - p)^{-n} + 1.$$

Left tail inequalities may be obtained in a similar fashion. When $np \to c$ for some constant $c \in (0, \infty)$ as $n \to \infty$, the bound is of the order at most $e^{-kt}$ for some positive constant $k$.

1 Introduction and main result

For some $n \in \{1, 2, \ldots, \}$ and $p \in (0, 1)$ let $K$ be the random graph on the vertices $V = \{1, 2, \ldots, n\}$, with the indicators $X_{vw}$ of the presence of edges between two unequal vertices $v$ and $w$ being independent Bernoulli random variables with success probability $p$, and $X_{vv} = 0$ for all $v \in V$. Recall that the degree of a vertex $v \in V$ is the number of edges incident on $v$,

$$d(v) = \sum_{w \in V} X_{vw}.$$

Many authors have studied the distribution of

$$Y = \sum_{v \in V} 1(d(v) = d)$$

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counting the number of vertices $v$ of $K$ with degree $d(v) = d$ for some fixed $d$. In this paper we derive upper bounds, for fixed $n$, on the distribution function of the number of isolated vertices of $K$, that is, (2) for the case $d = 0$ where $Y$ counts the number of vertices having no incident edges.

For $d$ in general, and $p = p_n$, depending on $n$, previously in [7] the asymptotic normality of $Y$ was shown when $n^{(d+1)/d}p_n \to \infty$ and $np_n \to 0$, or $np_n \to \infty$ and $np_n - \log n - d \log \log n \to -\infty$; see also [10] and [3]. Asymptotic normality of $Y$ when $np_n - c \to 0$, was obtained by [2]. The size bias coupling considered here used was in [6] to study the rate of convergence to the multivariate normal distribution for a vector whose components count the number of vertices of some fixed degrees. In [8], the mean $\mu$ and variance $\sigma^2$ of $Y$ for the particular case $d = 0$ are computed as

$$
\mu = n(1-p)^n \quad \text{and} \quad \sigma^2 = n(1-p)^{n-1}(1+np(1-p)^{n-2}-(1-p)^{n-2}).
$$

(3)

In the same paper, Kolmogorov distance bounds to the normal were obtained and asymptotic normality shown when

$$
n^2p \to \infty \quad \text{and} \quad np - \log(n) \to -\infty.
$$

O’Connell [9] showed that an asymptotic large deviation principle holds for $Y$. Raič [11] obtained nonuniform large deviation bounds in some generality, for random variables $W$ with $E(W) = 0$ and $\text{Var}(W) = 1$, of the form

$$
P(W \geq t) \leq e^{t\beta(t)/6}(1+Q(t)\beta(t)) \quad \text{for all } t \geq 0
$$

(4)

where $\Phi(t)$ denotes the distribution function of a standard normal variate and $Q(t)$ is a quadratic in $t$. Although in general the expression for $\beta(t)$ is not simple, when $W$ equals $Y$ properly standardized and $np \to c$ as $n \to \infty$, (4) holds for all $n$ sufficiently large with

$$
\beta(t) = \frac{C_1}{\sqrt{n}} \exp \left( \frac{C_2t}{\sqrt{n}} + C_3(e^{C_4t/\sqrt{n}} - 1) \right)
$$

for some constants $C_1, C_2, C_3$ and $C_4$. For $t$ of order $n^{1/2}$, for instance, the function $\beta(t)$ will be small as $n \to \infty$, allowing an approximation of the deviation probability $P(W \geq t)$ by the normal, to within some factors.

Theorem 1.1 below, by contrast, produces a non-asymptotic, explicit bound, that is, it does not require any relation between $n$ and $p$ and is satisfied for all $n$. Moreover by (9) and (7), the bound is of order $e^{-\lambda t^2}$ over some range of $t$, and of worst case order $e^{-\rho t}$, for the right tail, and $e^{-\eta t^2}$ for the left tail, where $\lambda, \rho$ and $\eta$ are explicit, with the bound holding for all $t \in \mathbb{R}$.

**Theorem 1.1.** For $n \in \{1, 2, \ldots\}$ and $p \in (0, 1)$ let $K$ denote the random graph on $n$ vertices where each edge is present with probability $p$, independently of all other edges, and let $Y$ denote the number of isolated vertices in $K$. Then for all $t > 0$,

$$
P \left( \frac{Y - \mu}{\sigma} \geq t \right) \leq \inf_{\theta \geq 0} \exp(-\theta t + H(\theta)) \quad \text{where} \quad H(\theta) = \frac{\mu}{2\sigma^2} \int_0^\theta \gamma_s ds,
$$

(5)

with the mean $\mu$ and variance $\sigma^2$ of $Y$ given in (3), and

$$
\gamma_s = 2e^{2s} \left( 1 + \frac{pe^s}{1-p} \right)^n + \beta + 1 \quad \text{where} \quad \beta = (1-p)^n.
$$

(6)

For the left tail, for all $t > 0$,

$$
P \left( \frac{Y - \mu}{\sigma} \leq -t \right) \leq \exp \left( -\frac{t^2}{2} \frac{\sigma^2}{\mu(\beta+1)} \right).
$$

(7)
Recall that for a nonnegative random variable $Y$ with finite, nonzero mean $\mu$, the size bias distribution of $Y$ is given by the law of a variable $Y^*$ satisfying
\[
E[Yf(Y)] = \mu E[f(Y^*)]
\]
for all $f$ for which the expectations above exist. The main tool used in proving Theorem 1.1 is size bias coupling, that is, constructing $Y$ and $Y^*$, having the $Y$-size biased distribution, on the same space. In [4] and [5], size bias couplings were used to prove concentration of measure inequalities when $|Y^* - Y|$ can be almost surely bounded by a constant independent of the problem size, the number of vertices in the present case. Here, when $Y$ is the number of isolated vertices of $K$, we consider a coupling of $Y$ to $Y^*$ with the $Y$-size bias distribution where this boundedness condition is violated. Unlike the theorem used in [4] and [5], which can be applied to a wide variety of situations under a bounded coupling assumption, it seems that cases where the coupling is unbounded, such as the one we consider here, need application specific treatment, and cannot be handled by one single general result.

**Remark 1.1.** Useful bounds for the minimization in (5) may be obtained by restricting to $\theta \in [0, \theta_0]$ for some $\theta_0$. In this case, as $\gamma_s$ is an increasing function of $s$, we have
\[
H(\theta) \leq \frac{\mu}{4\sigma^2} \gamma_s \theta^2, \quad \text{for } \theta \in [0, \theta_0].
\]
The quadratic $-\theta t + \mu \gamma_0 \theta^2/(4\sigma^2)$ in $\theta$ is minimized at $\theta = 2t\sigma^2/(\mu \gamma_0)$. When this value falls in $[0, \theta_0]$ we obtain the first bound in (9), and setting $\theta = \theta_0$ yields the second, thus,
\[
P\left(\frac{Y - \mu}{\sigma} > t\right) \leq \begin{cases} 
\exp(-\gamma_0^2 \theta^2 / (4\mu^2 \gamma_0^2)) & \text{for } t \in [0, \theta_0 \mu \gamma_0 / (2\sigma^2)] \\
\exp(- \theta_0 t + \mu \gamma_0 \theta^2 / (4\sigma^2)) & \text{for } t \in (\theta_0 \mu \gamma_0 / (2\sigma^2), \infty).
\end{cases}
\]

Though Theorem 1.1 is not an asymptotic result, when $np \to c$ as $n \to \infty$, (3) and (6) yield
\[
\frac{\sigma^2}{\mu} \to 1 + ce^{-c} - e^{-c}, \quad \beta \to e^c \quad \text{and} \quad \gamma_s \to 2e^{2s+ce^c} + e^c + 1 \quad \text{as } n \to \infty.
\]

Since $\lim_{n \to \infty} \gamma_s$ and $\lim_{n \to \infty} \mu/\sigma^2$ exist, the right tail probability is at most of the exponential order $e^{-\rho t}$ for some $\rho > 0$. Also, the left tail bound (7) in this asymptotic behaves as
\[
\lim_{n \to \infty} \exp\left(-\frac{t^2 \sigma^2}{2 \mu (\beta + 1)}\right) = \exp\left(-\frac{t^2}{2} \frac{1 + ce^{-c} - e^{-c}}{e^{-c} + 1}\right),
\]
that is, as $e^{-\eta t^2}$ with $\eta > 0$.

The paper is organized as follows. In Section 2 we review results leading to the construction of size biased couplings for sums of possibly dependent variables, and then apply it in Section 3 to the number $Y$ of isolated vertices; this construction first appeared in [6]. The proof of Theorem 1.1 is also given in Section 3.

### 2 Construction of size bias couplings

In this section we will review the discussion in [6] which gives a procedure for a construction of size bias couplings when $Y$ is a sum; the method has its roots in the work of Baldi et al. [1]. The construction depends on being able to size bias a collection of nonnegative random variables in a given coordinate, as described in Definition 2.1. Letting $F$ be the distribution of $Y$, first note that the characterization (8) of the size bias distribution $F^*$ is equivalent to the specification of $F^*$ by its Radon Nikodym derivative
\[
dF^*(x) = \frac{x}{\mu} dF(x).
\]
Definition 2.1. Let $A$ be an arbitrary index set and let $X = \{X_\alpha : \alpha \in A\}$ be a collection of nonnegative random variables with finite, nonzero expectations $EX_\alpha = \mu_\alpha$ and joint distribution $dF(x)$. For $\beta \in A$, we say that $X^\beta = \{X_\alpha^\beta : \alpha \in A\}$ has the $X$-size bias distribution in coordinate $\beta$ if $X^\beta$ has joint distribution

$$dP^\beta(x) = x_\beta dF(x)/\mu_\beta.$$  

Just as (10) is related to (8), the random vector $X^\beta$ has the $X$-size bias distribution in coordinate $\beta$ if and only if

$$E[X_\beta f(X)] = \mu_\beta E[f(X^\beta)]$$

for all functions $f$ for which these expectations exist.

Letting $f(X) = g(X_\beta)$ for some function $g$ one recovers (8), showing that the $\beta$th coordinate of $X^\beta$, that is, $X_\beta^\beta$, has the $X_\beta$-size bias distribution.

The factorization

$$dF(x) = P(X \in dx|X_\beta = x)P(X_\beta \in dx)$$

of the joint distribution of $X$ suggests the following way to construct $X$. First generate $X_\beta$, a variable with distribution $P(X_\beta \in dx)$. If $X_\beta = x$, then generate the remaining variates $\{X_\alpha, \alpha \neq \beta\}$ with distribution $P(X \in dx|X_\beta = x)$. Similarly, the factorization

$$dP^\beta(x) = x_\beta dF(x)/\mu_\beta = P(X \in dx|X_\beta = x) x_\beta P(X_\beta \in dx)/\mu_\beta = P(X \in dx|X_\beta = x)P(X_\beta^\beta \in dx)$$  \hspace{1cm} (11)

suggests that to generate $X^\beta$ with distribution $dP^\beta(x)$, one may first generate a variable $X_\beta^\beta$ with the $X_\beta$-size bias distribution, then, when $X_\beta^\beta = x$, generate the remaining variables according to their original conditional distribution given that the $\beta$th coordinate takes on the value $x$.

Definition 2.1 and the following special case of a proposition from Section 2 of [6] will be applied in the subsequent constructions; the reader is referred there for the simple proof.

Proposition 2.1. Let $A$ be an arbitrary index set, and let $X = \{X_\alpha, \alpha \in A\}$ be a collection of nonnegative random variables with finite means. Let $Y = \sum_{\beta \in A} X_\beta$ and assume $\mu_A = EY$ is finite and positive. Let $X^\beta$ have the $X$-size biased distribution in coordinate $\beta$ as in Definition 2.1. Then, if $I$ is a random index, independent of $X$, taking values in $A$ with distribution

$$P(I = \beta) = \mu_\beta/\mu_A, \quad \beta \in A,$$

the variable $Y^* = \sum_{\alpha \in A} X_\alpha^I$ has the $Y$-sized biased distribution as in (8).

In our examples we use Proposition 2.1 and the random index $I$, and (11), to obtain $Y^*$ by first generating $X^I_\beta$ with the size bias distribution of $X_\beta$, then, if $I = \beta$ and $X^\beta_\beta = x$, generating $\{X_\alpha^\beta : \alpha \in A\setminus\{\beta\}\}$ according to the (original) conditional distribution $P(X_\alpha, \alpha \neq \beta|X_\beta = x)$.

3 Proof of Theorem 1.1

We now present the proof of Theorem 1.1.

Proof. We first construct a coupling of $Y^*$, having the $Y$-size bias distribution, to $Y$. Let $K$ be given, and let $Y$ be the number of isolated vertices in $K$. To size bias $Y$, first recall the representation of $Y$ as the sum (2) with $d = 0$. As the summands in (2) are exchangeable, the distribution of the random index $I$ in Proposition 2.1 is uniform. Hence, choose one of the $n$ vertices of $K$ uniformly. If the chosen vertex, say $V$, is already isolated, we do nothing and set $K^* = K$, as the remaining variables already have their conditional distribution given that $V$ is isolated. Otherwise obtain $K^*$ by deleting all the edges connected to $K$. By Proposition 2.1, the variable $Y^*$ counting the number of isolated vertices of $K^*$ has the $Y$-size biased distribution.
To derive the needed properties of this coupling, let \( N(v) \) be the set of neighbors of \( v \in \mathcal{V} \), and \( \mathcal{T} \) be the collection of isolated vertices of \( K \), that is, with \( d(v) \), the degree of \( v \), given in (1),
\[
N(v) = \{ w : X_{vw} = 1 \} \quad \text{and} \quad \mathcal{T} = \{ v : d(v) = 0 \}.
\]
Note that \( Y = |\mathcal{T}| \). Since all edges incident to the chosen \( V \) are removed in order to form \( K^* \), any neighbor of \( V \) which had degree one thus becomes isolated, and \( V \) also becomes isolated if it was not so earlier. As all other vertices are otherwise unaffected, as far as their being isolated or not, we have
\[
Y^* - Y = d_1(V) + 1(d(V) \neq 0) \quad \text{where} \quad d_1(V) = \sum_{w \in N(V)} 1(d(w) = 1). \tag{12}
\]
In particular the coupling is monotone, that is, \( Y^* \geq Y \). Since \( d_1(V) \leq d(V) \), (12) yields
\[
Y^* - Y \leq d(V) + 1. \tag{13}
\]
Now note that for real \( x \neq y \), the convexity of the exponential function implies
\[
e^{-y} - e^{-x} = \int_0^1 e^{ty+(1-t)x} dt \leq \int_0^1 (te^y + (1-t)e^x) dt = \frac{e^y + e^x}{2}. \tag{14}
\]
Using (14), that the coupling is monotone, and that \( Y \) is a function of \( \mathcal{T} \), for \( \theta \geq 0 \) we have
\[
E(e^{\theta Y^*} - e^{\theta Y}) \leq \frac{\theta}{2} E \left( (Y^* - Y)(e^{\theta Y^*} + e^{\theta Y}) \right) \\
= \frac{\theta}{2} E \left( \exp(\theta Y)(Y^* - Y) \left( \exp(\theta(Y^* - Y)) + 1 \right) \right) \\
= \frac{\theta}{2} E \left\{ \exp(\theta Y) \left( (Y^* - Y) \left( \exp(\theta(Y^* - Y)) + 1 \right) \right) \mathbb{1}(|T|) \right\}. \tag{15}
\]
Now using that \( Y^* = Y \) when \( V \in \mathcal{T} \), and (13), we have
\[
E((Y^* - Y) \left( \exp(\theta(Y^* - Y)) + 1 \right) \mathbb{1}(|T|)) \\
\leq E((d(V) + 1) \left( \exp(\theta(d(V) + 1)) + 1 \right) \mathbb{1}(V \notin \mathcal{T}) \mathbb{1}(|T|)) \\
\leq e^\theta E \left( \left( d(V) \left( e^{\theta d(V)} + e^{-\theta d(V)} \right) + 1 \right) \mathbb{1}(V \notin \mathcal{T}) \mathbb{1}(|T|) \right) + 1, \tag{16}
\]
where in the final inequality we have used that \( 1(V \notin \mathcal{T}) \leq d(V) 1(V \notin \mathcal{T}) \).

To derive a bound on the expectation in (16), first note that by conditioning we obtain
\[
P(d(V) = k, 1(V \notin \mathcal{T}) = 1|\mathcal{T}) = P(V \notin \mathcal{T})P(d(V) = k|\mathcal{T}, V \notin \mathcal{T}). \tag{17}
\]
Since \( V \) is chosen independently of \( K \), the distribution on the right hand side of (17) is a Binomial sum with number of trials equal to the number \( n - 1 - Y \) of non-isolated vertices other than \( V \), each having success probability \( p \), conditioned to be nonzero, that is,
\[
P(d(V) = k|\mathcal{T}, V \notin \mathcal{T}) = \begin{cases} (n-1-y)^{k-1} y^{n-1-k} \prod_{k=1}^{\gamma} (1-p^{n-k-y}) & \text{for } 1 \leq k \leq n-1-Y \\ 0 & \text{otherwise.} \end{cases} \tag{18}
\]
Using the conditional distribution (18) and (17), and dropping the factor \( P(V \notin \mathcal{T}) \) in the later to obtain an inequality, it can be easily verified that the first derivative of the conditional moment generating function of \( d(V) \) satisfies
\[
E(d(V) e^{\theta d(V)} 1(V \notin \mathcal{T}) |\mathcal{T}) \leq \frac{(n-1-y)(pe^\theta + 1-p)^{n-2-Y} pe^\theta}{1 - (1-p)^{n-1-y}}.
\]
Hence for Standardizing, we set
\[ m = (n - 1 - Y)p \xi^{n-2-Y} \geq (n - 1 - Y)p(1 - p)^n. \]
Hence, recalling \( \theta \geq 0, \)
\[
E(d(V)e^{\theta d(V)}1(V \notin T)|T) \leq \frac{(n - 1 - Y)(p e^\theta + 1 - p)^n p e^\theta}{1 - (1 - p)^{n-1-Y}} \leq \frac{(n - 1 - Y)(p e^\theta + 1 - p)^n p e^\theta}{(n - 1 - Y)p(1 - p)^n} = \alpha_{\theta} \text{ where } \alpha_{\theta} = e^\theta \left(1 + \frac{pe^\theta}{1 - p}\right)^n. \tag{19}
\]
Lastly, again applying (17) and (18) we may handle the second term in (16) by the inequality
\[
E(d(V)1(V \notin T)|T) \leq \frac{(n - 1 - Y)p}{1 - (1 - p)^{n-1-Y}} \leq \frac{(n - 1 - Y)p}{(n - 1 - Y)p(1 - p)^n} = \beta, \tag{20}
\]
with \( \beta \) as in (6).
Substituting inequalities (19) and (20) into (16) yields
\[
E ((Y^s - Y)(\exp(\theta(Y^s - Y)) + 1)|T) \leq \gamma_{\theta} \text{ where } \gamma_{\theta} = 2e^\theta \alpha_{\theta} + \beta + 1. \tag{21}
\]
Now, by (15) we have that
\[
E(e^{\theta Y^s} - e^{\theta Y}) \leq \frac{\theta \gamma_{\theta}}{2} E(e^{\theta Y}) \text{ for all } \theta \geq 0. \tag{22}
\]
Letting \( m(\theta) = E(e^{\theta Y}) \) and using (8) and (22) we obtain
\[
m'(\theta) = E(Y e^{\theta Y}) = \mu E(e^{\theta Y^s}) \leq \mu \left(1 + \frac{\theta \gamma_{\theta}}{2}\right) m(\theta). \tag{23}
\]
Standardizing, we set
\[
M(\theta) = E(\exp(\theta(Y - \mu)/\sigma)) = e^{-\theta \mu / \sigma} m(\theta / \sigma), \tag{24}
\]
and now by differentiating and applying (23), we obtain
\[
M'(\theta) = \frac{1}{\sigma} e^{-\theta \mu / \sigma} m'(\theta / \sigma) - \frac{\mu}{\sigma} e^{-\theta \mu / \sigma} m(\theta / \sigma) \leq \frac{\mu}{\sigma} e^{-\theta \mu / \sigma} \left(1 + \frac{\theta \gamma_{\theta}}{2\sigma}\right) m(\theta / \sigma) - \frac{\mu}{\sigma} e^{-\theta \mu / \sigma} m(\theta / \sigma) = e^{-\theta \mu / \sigma} \frac{\mu \theta \gamma_{\theta}}{2\sigma^2} m(\theta / \sigma) = \frac{\mu \theta \gamma_{\theta}}{2\sigma^2} M(\theta).
\]
Since \( M(0) = 1 \), integrating \( M'(s)/M(s) \) over \([0, \theta]\) yields the bound
\[
\log(M(\theta)) \leq H(\theta), \text{ or that } M(\theta) \leq \exp(H(\theta)) \text{ where } H(\theta) = \frac{\mu}{2\sigma^2} \int_0^\theta s \gamma_s ds.
\]
Hence for \( t \geq 0, \)
\[
P \left( \frac{Y - \mu}{\sigma} \geq t \right) \leq P \left( \exp \left( \frac{\theta(Y - \mu)}{\sigma} \right) \geq e^{\theta t} \right) \leq e^{-\theta t} M(\theta) \leq \exp(-\theta t + H(\theta)).
\]
As the inequality holds for all $\theta \geq 0$, it holds for the $\theta$ achieving the minimal value, proving (5).

To demonstrate the left tail bound let $\theta < 0$. Since $Y_s \geq Y$ and $\theta < 0$, using (14) and (13), and recalling $Y_s = Y$ when $V \in T$, we obtain

\[
E(e^{\theta Y} - e^{\theta Y^*}) \leq \frac{|\theta|}{2} E \left( (e^{\theta Y} + e^{\theta Y^*})(Y^* - Y) \right) \\
\leq |\theta| E(e^{\theta Y}(Y^* - Y)) \\
= |\theta| E(e^{\theta Y} E(Y^* - Y|T)) \\
\leq |\theta| E(e^{\theta Y} E((d(V) + 1)(V \not\in T)|T)).
\]

Now (20) yields

\[
E(e^{\theta Y} - e^{\theta Y^*}) \leq (\beta + 1)|\theta| E(e^{\theta Y}),
\]

and therefore

\[
m'(\theta) = \mu E(e^{\theta Y^*}) \geq \mu (1 + (\beta + 1)\theta) m(\theta).
\]

Again with $M(\theta)$ as in (24),

\[
M'(\theta) = \frac{1}{\sigma} e^{-\theta\mu/\sigma} m'(\theta/\sigma) - \frac{\mu}{\sigma} e^{-\theta\mu/\sigma} m(\theta/\sigma) \\
\geq \frac{\mu}{\sigma} e^{-\theta\mu/\sigma} ((1 + (\beta + 1)\theta/\sigma) m(\theta/\sigma)) - \frac{\mu}{\sigma} e^{-\theta\mu/\sigma} m(\theta/\sigma) \\
= \frac{\mu(\beta + 1)\theta}{\sigma^2} M(\theta).
\]

Dividing by $M(\theta)$ and integrating over $[\theta, 0]$ yields

\[
\log(M(\theta)) \leq \frac{\mu(\beta + 1)\theta^2}{2\sigma^2}.
\]

The inequality in (25) implies that for all $t > 0$ and $\theta < 0$,

\[
P \left( Y - \frac{\mu}{\sigma} \leq -t \right) \leq \exp \left( \theta t + \frac{\mu(\beta + 1)\theta^2}{2\sigma^2} \right).
\]

Taking $\theta = -t\sigma^2/(\mu(\beta + 1))$ we obtain (7).

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References


