Structured Lattice Codes for $2 \times 2 \times 2$ MIMO Interference Channel

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• All nodes have $M$ multiple antennas and each node has the power-constraint of $P$

• The channel matrices $(F_{\ell k}, G_{\ell k})$ are drawn i.i.d. according to a continuous distribution and constant over the whole coding block of length $n$

• **Symmetric rate**, i.e., all messages have the same rate of $R$ (Sum-rate is $(2M - 1)R$)
This model has been extensively studied, being one of the fundamental building blocks for general two-flow networks

- **DoF = 4/3** [Cadambe-Jafar IT2009], interference alignment by each hop viewing X-channel

- **DoF = 2** [Gou-Jafar-Jeon-Chung IT2011], aligned interference neutralization by combining interference alignment and interference neutralization

- For the MIMO case ($M$ antennas)
  - The sum-DoF of $(2M - 1)$ was achieved using aligned interference neutralization (Optimal when $M \to \infty$)
Our Contributions

- We present Precoded CoF with Channel Integer Alignment (PCoF with CIA)
  - Based on lattice codes
  - Achieve the same sum-DoF of $(2M - 1)$
  - Focus on the performance of finite-SNR by exploiting the algebraic structure of lattices
Phase I: Convert the two-user Gaussian IC into a deterministic Finite-Field IC
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- Signal Alignment Scheme [Gou-Jafar-Jeon-Chung IT2011]
  - also used for aligned interference neutralization
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  - a generalization of zero-forcing receiver
PCoF with CIA

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PCoF with CIA: Encoding

- Channel Integer Alignment (CIA):
  - Sources 1 and 2 choose the precoding matrices $V_1$ and $V_2$ such that
    \[
    \begin{bmatrix}
      F_{k1} V_1 & F_{k2} V_2
    \end{bmatrix} = H_k C_k : \text{Alignment Condition}
    \]

    where $H_k \in \mathbb{C}^{M \times M}$ and $C_k \in \mathbb{Z}[j]^{M \times 2M}$

- Power-Penalty:

  \[
  \text{SNR} = \min \left\{ \frac{P}{\text{tr}(V_1 V_1^H)}, \frac{P}{\text{tr}(V_2 V_2^H)} \right\}
  \]

[Problem] We want to choose the $V_1$ and $V_2$ that satisfy the alignment condition and minimize the power penalty (Not manageable).
[Separate Approach]

- Determine the precoding matrices $V_1$ and $V_2$ to satisfy the alignment condition [Gou-Jafar-Jeon-Chung IT2011]:

  $$
  \begin{align*}
  v_{1,\ell+1} &= \left( F_{11}^{-1} F_{12} F_{22}^{-1} F_{21} \right)^{\ell} v_{1,1} \\
  v_{2,\ell} &= \left( F_{22}^{-1} F_{21} F_{11}^{-1} F_{12}^{-1} \right)^{\ell-1} F_{22}^{-1} F_{21} v_{1,1} \\
  v_{1,1} &= E1
  \end{align*}
  $$

  where the columns of $E$ consist of eigenvectors of $(F_{11}^{-1} F_{12} F_{22}^{-1} F_{21})$

- In order to minimize the power-penalty, we use the concept of Integer-Forcing Beamforming (IFB) [Hong-Caire ISIT2012]
For any full-rank integer matrix $A$, the followings are equivalent (under mod-$\Lambda$ channel):

\[ w_1 \rightarrow \text{Lattice Encoding} \rightarrow t_1 \rightarrow \text{Precoding (V)} \]

\[ \vdots \rightarrow \vdots \rightarrow \vdots \rightarrow \vdots \]

\[ w_M \rightarrow \text{Lattice Encoding} \rightarrow t_M \rightarrow \text{Precoding (V)} \]

⇒ Optimize $A$ to minimize the power-penalty $\text{tr}(VAV^H) \leq \text{tr}(VV^H)$

"No power-penalty"

\[ w_1 \rightarrow \text{Precoding over Finite-Field} \rightarrow w'_1 \rightarrow \text{Lattice Encoding} \rightarrow t'_1 \rightarrow \text{Precoding (VA)} \]

\[ \vdots \rightarrow \vdots \rightarrow \vdots \rightarrow \vdots \]

\[ w_M \rightarrow \text{Precoding over Finite-Field} \rightarrow w'_M \rightarrow \text{Lattice Encoding} \rightarrow t'_M \rightarrow \text{Precoding (VA)} \]
Decoding at Relays:

- At relay 1:

\[
Y_1 = F_{11} V_1 T_1 + F_{12} V_2 T_2 + Z_1
\]

where $V_1$ and $V_2$ satisfy the alignment condition
Decoding at Relays:

- At relay 1:

\[
\mathbf{Y}_1 = \mathbf{F}_{11} \mathbf{V}_1 \mathbf{T}_1 + \mathbf{F}_{12} \mathbf{V}_2 \mathbf{T}_2 + \mathbf{Z}_1
\]

\[
= \mathbf{H}_1 \mathbf{C}_1 \begin{bmatrix} \mathbf{T}_1 \\ \mathbf{T}_2 \end{bmatrix} + \mathbf{Z}_1
\]

where we have \( M \) lattice codewords

\[
\mathbf{T}'_1 = \mathbf{C}_1 \begin{bmatrix} \mathbf{T}_1 \\ \mathbf{T}_2 \end{bmatrix}
\]
PCoF with CIA: Decoding

$[M \times M \text{ MIMO Channel}]:$

- At relay 1:

\[
\mathbf{Y}_1 = \mathbf{F}_{11} \mathbf{V}_1 \mathbf{T}_1 + \mathbf{F}_{12} \mathbf{V}_2 \mathbf{T}_2 + \mathbf{Z}_1 = \mathbf{H}_1 \mathbf{T}'_1 + \mathbf{Z}_1
\]

where $\mathbf{H}_1 \in \mathbb{C}^{M \times M}$
[\( M \times M \) MIMO Channel]:

- At relay 1:

\[
Y_1 = F_{11}V_1T_1 + F_{12}V_2T_2 + Z_1 = H_1T'_1 + Z_1
\]

where \( H_1 \in \mathbb{C}^{M \times M} \)

- At relay 2:

\[
Y_2 = H_2T'_2 + Z_2
\]

where \( H_2 \in \mathbb{C}^{M \times M} \)
Each relay $k$ can decode $M$ lattice codewords $\mathbf{T}_k'$ using Zero-Forcing Receiver:

$$\hat{\mathbf{Y}}_1 = H_k^{-1}\mathbf{Y}_k = \mathbf{T}_k' + H_k^{-1}\mathbf{Z}_k$$

We can do better using Integer-Forcing Receiver (IFR):

$$\hat{\mathbf{Y}}_1 = B_k H_k^{-1}\mathbf{Y}_k = B_k \mathbf{T}_k' + B_k H_k^{-1}\mathbf{Z}_k$$

- Decode $M$ lattice codewords $B_k \mathbf{T}_k'$
- Using matrix inversion with $B_k^{-1}$, we can get $\mathbf{T}_k'$
Integer-Forcing Receiver

- Each relay $k$ can decode $M$ lattice codewords $\mathbf{T}'_k$ using Integer-Forcing Receiver:
  \[
  \hat{\mathbf{Y}}_1 = \mathbf{B}_k \mathbf{H}_k^{-1} \mathbf{Y}_k = \mathbf{B}_k \mathbf{T}'_k + \mathbf{B}_k \mathbf{H}_k^{-1} \mathbf{Z}_k
  \]

- Achievable Rate [Zhan-Nazer-Erez-Gastpar ISIT2010]:
  \[
  R(\mathbf{B}_k) = \min_{k=1,2} \left\{ \log^+ \left( \frac{\text{SNR}}{\sigma_{\text{eff},k,\ell}^2} \right) : \ell = 1, \ldots, M \right\}
  \]
  where
  \[
  \sigma_{\text{eff},k,\ell}^2 = \| (\mathbf{H}_k)^{-1} \mathbf{b}_{k,\ell} \|^2.
  \]

- Optimize over full-rank integer matrix $\mathbf{B}_k \in \mathbb{Z}[j]^{M \times M}$ to minimize the effective noise
Deterministic $2 \times 2 \times 2$ Finite-Field IC

- Each relay $k$ knows the $M$ lattice codewords $T'_k$ with $t'_{k,\ell} = f(u_{k,\ell})$ for $\ell = 1, \ldots, M$

- Using the *linearity* of lattice encoding:

\[
\begin{bmatrix}
  u_{k,1} \\
  \vdots \\
  u_{k,M}
\end{bmatrix}
= g^{-1}([C_k] \mod p\mathbb{Z}[j])
\begin{bmatrix}
  w_{1,1} \\
  \vdots \\
  w_{1,M} \\
  w_{2,1} \\
  \vdots \\
  w_{2,M-1}
\end{bmatrix}
\]

- Define a *deterministic* finite-field IC with system matrix (same for the second-hop):

\[
Q = \begin{bmatrix}
  g^{-1}([C_1] \mod p\mathbb{Z}[j]) \\
  g^{-1}([C_2] \mod p\mathbb{Z}[j])
\end{bmatrix}
\in \mathbb{F}_p^{(2M-1) \times (2M-1)}
Define a deterministic finite-field IC with system matrix (same for the second-hop):

\[ Q = \begin{bmatrix} I_{M \times M} & Q_{12} \\ Q_{21} & I_{(M-1) \times (M-1)} \end{bmatrix} \]

where

\[ Q_{12} = \begin{bmatrix} 0_{1 \times (M-1)} \\ I_{(M-1) \times (M-1)} \end{bmatrix} \text{ and } Q_{21} = \begin{bmatrix} I_{(M-1) \times (M-1)} & 0_{(M-1) \times 1} \end{bmatrix} \]

⇒ This matrix is fixed and independent of channel matrices
**Phase II:** We propose a linear precoding scheme over finite-field in order to eliminate the end-to-end interferences

\[ Q \begin{bmatrix} M_1 & M_2 \end{bmatrix} Q = "\text{diagonal matrix}" \]
Phase 2: We propose a linear precoding scheme over finite-field in order to eliminate the end-to-end interferences

\[ Q \begin{bmatrix} M_1 & M_2 \end{bmatrix} Q = \text{"diagonal matrix"} \]
Deterministic $2 \times 2 \times 2$ Finite-Field IC

- $M = 2$, $Q = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$, $M_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $M_2 = [-1]$
For general $M \geq 2$,

**Lemma 1.** *Choosing precoding matrices (over finite-field) $M_1$ and $M_2$ as*

$$
M_1 = (I_{M \times M} - Q_{12} Q_{21})^{-1} \\
M_2 = -(I_{(M-1) \times (M-1)} - Q_{21} Q_{12})^{-1}
$$

*the end-to-end system matrix becomes a diagonal matrix:*

$$
Q \begin{bmatrix}
M_1 \\
M_2
\end{bmatrix} Q = 
\begin{bmatrix}
I_{M \times M} & \\
& -I_{(M-1) \times (M-1)}
\end{bmatrix}
$$
Optimization of Integer Matrices $A$ and $B$

- IFB (Minimize the power-penalty)

$$\text{minimize} \quad \text{tr}(VAA^HV) = \sum_{\ell} \|Va_{\ell}\|^2$$

subject to $A \in \mathbb{Z}[j]^{M \times M}$ is full-rank

- Equivalent to find the shortest set of linearly independent, full-rank vectors for the lattice generated by $V \Rightarrow "\text{Shortest Independent Vector Problem (SIVP)}"$

- Several polynomial-time algorithms have been developed, such as LLL algorithm
Optimization of Integer Matrices $A$ and $B$

- IFR (Minimize the effective-noise)

\[
\text{minimize} \quad \max \{ \| H^{-1} b_\ell \|^2 : \ell = 1, \ldots, M \}
\]

subject to $B \in \mathbb{Z}[j]^{M \times M}$ is full-rank

- Equivalent to find the shortest set of linearly independent, full-rank vectors for the lattice generated by $H^{-1}$ ⇒ ”Shortest Independent Vector Problem (SIVP)”

- Several polynomial-time algorithms have been developed, such as LLL algorithm
Numerical Results

- All nodes have 2 transmit/receiver antennas (i.e., $M = 2$)

- Channel realizations with i.i.d. Rayleigh fading $\sim \mathcal{CN}(0, 1)$

- Comparing with Time-Sharing (IFR is used for each $M \times M$ MIMO) with $2P$ transmission power

- Used LLL algorithm to determine the integer matrices $A$ and $B$
Numerical Results

![Graph showing numerical results]

- Average symmetric sum rates (bits per channel use)
- Time Sharing
- PCoF with CIA (w/o opt)
- PCoF with CIA
- 3 DoF
- 2 DoF
Thank You!