A TOURNAMENT DISTRIBUTION

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Let us say that a tournament consists of two teams, A and B, who play each other at each round and A wins with probability p and B wins with probability $1-p$ (i.e. there is no tie). The teams will play against one another until one team has accumulated n wins. Let $N =$ the number of games played in the tournament. Compute the distribution of N.

Solution. Consider the case $n = 1$. Then we have either A or B wins, and so $N = 1$ with probability 1. In this case N is called a *trivial* random variable since it only assumes one value.

The case $n = 2$ is only slightly more complicated, and one can easily enumerate all possibilities; similarly with $n = 3$. However, for $n \geq 4$ the enumeration rapidly becomes impractical, and we must come up with a more general solution.

Let us think of the extreme cases for arbitrary n. If one team wins all of the games then the smallest value of N is n. It could also happen that one team wins $n-1$ and the other team wins n, so N is at most $2n-1$.

Let us consider what happens when $N = k$ for $n \leq k \leq 2n-1$. Either A or B has a total of n wins, so let us assume that it is A for now. We then have that A won exactly n of the k rounds and B won exactly $k-n$ of the rounds. Does it matter in what order the rounds were won? The answer is YES!

We must be careful in that the last round must be a win for A, but the previous $k-1$ rounds CAN in fact be in any order. So the better answer is that the order does NOT matter in the first $k-1$ rounds, in which case A won $n-1$ of them and B won $n-k$ of them; then A wins the final (k^{th}) round.

This explanation is easily seen to be solvable using a Binomial distribution, since the order of the first $k-1$ rounds doesn't matter and we may consider A winning to be a "success." We then obtain

$$\begin{aligned}
P(N=k) &= P(A \text{ wins tournament in k rounds}) + P(B \text{ wins tournament in k rounds}) \\
&= \left[\binom{k-1}{n-1}p^{n-1}(1-p)^{k-n}\right]p + \left[\binom{k-1}{n-1}(1-p)^{n-1}p^{k-n}\right](1-p) \\
&= \left[\binom{k-1}{n-1}p^{n}(1-p)^{k-n}\right] + \left[\binom{k-1}{n-1}(1-p)^{n}p^{k-n}\right].
\end{aligned}$$

And again, as we mentioned before, $N \in \{n, n+1, \ldots, 2n-1\}$ so this formula applies for $n \leq k \leq 2n-1$. Note the similarity to the negative binomial distribution HOWEVER in the negative binomial distribution we are allowed an indefinite amount of failures until we obtain n successes.

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Writing down a formula for the expected value is easy, but simplifying the resulting sum is hard, and I don't know of any simple closed form solution, even for $p = 1/2$ (although I may not have looked hard enough).

One final comment is that since we know that $P(N = k)$ is the distribution of a random variable, we have that

$$\sum_{k=n}^{2n-1} P(N = k) = 1,$$

so in particular using a probabilistic argument we have provided a proof for the identity

$$\sum_{k=n}^{2n-1} \left(\left[\binom{k-1}{n-1} p^n (1-p)^{k-n} \right] + \left[\binom{k-1}{n-1} (1-p)^n p^{k-n} \right] \right) = 1.$$