How to obtain a fair coin from a p-coin

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This article will demonstrate a method for generating results for an unbiased coin using one that is biased[1]. There is no such physical thing as a perfectly unbiased coin, however, there does exist a procedure that will give a theoretically unbiased coin. Let us see how.

First, a p-coin is a coin that is heads with probability p (and tails with probability $1-p$), and this probability remains constant no matter how many times we flip the coin. We would like to simulate a fair coin ($p = 1/2$) using an arbitrary p-coin.

Denote by H the event of a heads and T that of tails. Consider the following procedure:

I flip the p-coin *twice*. If the outcome is HT I record heads. If the outcome is TH I record an answer of tails. If the outcome is HH or TT then I restart and flip twice again until I obtain either a TH or a HT.

Problem 1 *Show that the probability of a heads using the above procedure is 1/2.*

Solution 1

$$\begin{aligned} P(heads) &= P(HT) + P((HH \cup TT) \cap HT) + \ldots \\ &= \sum_{k=0}^{\infty} p(1-p)(p^2 + (1-p)^2)^k \\ &= \frac{p(1-p)}{1 - (p^2 + (1-p)^2)} \\ &= \frac{1}{2}. \end{aligned}$$

Problem 2 *Find the expected number of trials needed before an outcome of HT or TH is recorded (i.e. how many sets of two coin-flips are required?)*

Solution 2 *Let X = the number of trials needed before the procedure ends. Then we have*

$$P(X = k) = (p^2 + (1-p)^2)^{k-1}(2p(1-p))$$

[1]This problem is taken from Introduction to Probability, Vol. 1, Feller

Since the first term is the probability of getting HH or TT $k-1$ times followed by either a HT or TH at the k^{th} trial. We then have

$$
\begin{aligned}
\mathbb{E}\,X &= \sum_{k=1}^{\infty} k(p^2+(1-p)^2)^{k-1} 2p(1-p) \\
&= 2p(1-p)\sum_{k=1}^{\infty} k(p^2+(1-p)^2)^{k-1} \\
&= \frac{2p(1-p)}{(1-(p^2+(1-p)^2))^2} \\
&= \frac{1}{2p(1-p)}.
\end{aligned}
$$

Exercise 1 *Graph $\mathbb{E}\,X$ as a function of p. For which values of p is $\mathbb{E}\,X$ a minimum/maximum? Does this agree with your intuition?*