MATH 407 FALL 2009

QUIZ 7

1. The joint density function of two random variables X and Y is

$$f(x,y) = c(y^2 + x^3), \quad -y \le x \le y, \quad 0 \le y \le 1.$$

Find (a) the value of c, (b) the marginal distribution of Y, and (c) the expected value of Y.

a) c satisfies

$$1 = \int_0^1 \int_{-y}^{y} c(y^2 + x^3)dxdy,$$

so

$$\begin{aligned}\int_0^1 \int_{-y}^{y} c(y^2 + x^3)dxdy &= c\int_0^1 2y^3 dy \\ &= c/2,\end{aligned}$$

hence $c = 2$.

b)

$$\begin{aligned}f_Y(y) &= 2\int_{-y}^{y} y^2 + x^3 dx \\ &= 2(2y^3) \\ &= 4y^3.\end{aligned}$$

c)

$$\begin{aligned}\mathbb{E}Y &= \int_0^1 f_Y(y)dy \\ &= \int_0^1 4y^3 dy \\ &= 1.\end{aligned}$$

2. Two students take the SATs (which have integer-valued numerical scores between 0 and 1600) and their scores are approximately normal. Student A's score has expected value 1300 and standard deviation 100, and Student B's score has expected value 1400 and standard deviation 50. Assuming the scores are independent, compute the probability that, in one exam, Student A scores higher than Student B. (Hint: You may leave your answer in the form of $P(a < Z < b)$ where Z is the standard normal distribution)

The probability we are after is $P(A > B)$, which can be written as

$$\begin{aligned} P(A > B) &= P(A - B > 0) \\ &= P(A - B \geq 1) \\ &\approx P(Y \geq 0.5) \\ &= P\left(\frac{Y + 100}{\sqrt{50^2 + 100^2}} \geq \frac{100.5}{\sqrt{50^2 + 100^2}}\right) \\ &= P\left(Z \geq \frac{100.5}{\sqrt{50^2 + 100^2}}\right), \end{aligned}$$

where Y is a normal random variable with mean -100 and standard deviation $\sqrt{50^2 + 100^2}$ that approximates the distribution of $A - B$.