MATH 126 SPRING 2011, QUIZ 8

1. Find the hydrostatic force on a thin circular plate with radius 5 ft if the plate is submerged in water 10 ft deep (i.e., the bottom of the plate is touching the bottom of the container of water).

   The picture looks like a circle circumscribed into a rectangle of height 10. To find the total force, we start with setting coordinate axes. There are many choices but probably the most intuitive choice will be going through the center of the circle, with positive x going towards the surface of the water (so $x = 1$ means we are at a depth of 4 feet). This way the upper semi-circle has length given by $l(x) = 2\sqrt{25 - x^2}$, and we need only worry about the depth function.

   At $x = 0$, we are at a depth of 5 feet, and when $x = +5$, we have a depth of 0. Our depth function is then $d(x) = 5 - x$.

   Putting this all together, we have

$$F = \int P(x)A(x) = \int \rho g d(x) l(x) dx = \rho g \int (5 - x) 2\sqrt{25 - x^2} dx.$$

   Finally, the bounds of integration are -5 and 5, hence

$$F = \int_{-5}^{5} \rho g (5 - x) 2\sqrt{25 - x^2} dx = 2\rho g \left(\int_{-5}^{5} 5\sqrt{25 - x^2} dx - \int_{-5}^{5} x\sqrt{25 - x^2} dx \right).$$

   The first integral is 5 times half the area of a circle with radius 5, and the second integral is 0 by symmetry (or a simple u-substitution can evaluate the integral directly).

$$F = 2\rho g 5\pi 25/2 = \rho g 5^3 = 125\rho g.$$

2. Find the centroid of the region bounded by $y = \sin x$, $y = \cos x$, $x = 0$, $x = \pi/4$. (Hint: $\cos^2 x = (1 + \cos(2x))/2, \sin^2 x = (1 - \cos(2x))/2$.)

$$\begin{aligned}
\overline{x} &= \frac{M_y}{\rho A} \\
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M_y &= \rho \int_a^b x f(x) dx = \rho \int_0^{\pi/4} x(\cos(x) - \sin(x)) dx \\
M_x &= \rho/2 \int_a^b f(x)^2 dx = \rho/2 \int_0^{\pi/4} (\cos(x)^2 - \sin(x)^2) dx \\
A &= \int_a^b f(x) dx = \int_0^{\pi/4} (\cos(x) - \sin(x)) dx.
\end{aligned}$$

3. Find the center of mass of a lamina with density $\rho = 2$ and whose shape is a quarter circle with radius r in quadrant 3 (i.e., lower left part of a circle with radius r).

First, the area of the quarter-circle is $\pi r^2/4$. The function that we need to integrate is then $f(x) = -\sqrt{r^2 - x^2}$ since we are looking at the bottom half, and the bounds of integration are $-r$ to 0. We then have

$$M_x = \rho/2 \int_{-r}^{0} (r^2 - x^2)dx$$

$$M_y = \rho \int_{-r}^{0} x\sqrt{r^2 - x^2}dx$$

$$\overline{x} = \frac{\int_{-r}^{0} x\sqrt{r^2 - x^2}dx}{\pi r^2/4}$$

$$\overline{y} = \frac{\frac{1}{2}\int_{-r}^{0} r^2 - x^2 dx}{\pi r^2/4}$$

The rest follows by computation.