MATH 126 SPRING 2011, QUIZ 6

1. Determine whether each integral is convergent or divergent. Evaluate those that are convergent.

   a. $\int_1^\infty \frac{1}{(3x+1)^2}dx$

   There is a point of discontinuity at $x = 1/3$, but this is not in our region of integration, so we do not need to split up the integral.

$$\begin{aligned}
\int_1^\infty \frac{1}{(3x+1)^2}dx &= \lim_{a\to\infty} \int_1^a \frac{1}{(3x+1)^2}dx \\
u = 3x+1 \quad &, \quad du = 3dx \\
&= \lim_{a\to\infty} \frac{1}{3}\int_4^a u^{-2}du \\
&= \lim_{a\to\infty} \frac{1}{3}(-u^{-1}|_4^a) \\
&= \lim_{a\to\infty} \frac{1}{3}\left(\frac{1}{4}-\frac{1}{a}\right) \\
&= \frac{1}{12} < \infty
\end{aligned}$$

   Therefore the integral is CONVERGENT.

   b. $\int_{2\pi}^\infty \sin\theta d\theta$

$$\begin{aligned}
\int_{2\pi}^\infty \sin\theta d\theta &= \lim_{a\to\infty}\int_{2\pi}^a \sin\theta d\theta \\
&= \lim_{a\to\infty} -\cos\theta|_{2\pi}^a \\
&= \lim_{a\to\infty} \cos 2\pi - \cos a \\
&= \lim_{a\to\infty} 1-\cos a,
\end{aligned}$$

   but $\lim_{a\to\infty}\cos a$ does not exist, hence the integral is DIVERGENT.

   c. $\int_0^2 z^2\ln z dz$

$$\begin{aligned}
\int_0^2 z^2 \ln z dz &= \lim_{a\to 0^+}\int_a^2 z^2\ln z dz \\
u = \ln z \quad &, \quad dv = z^2 \\
du = 1/z dz \quad &, \quad v = z^3/3 \\
&= \lim_{a\to 0^+} \frac{1}{3}z^3\ln z|_a^2 - \int_a^2 \frac{1}{3}z^2 dz \\
&= \lim_{a\to 0^+} \frac{1}{3}(8\ln 2 - a^3\ln a) - \frac{1}{9}(8-a^3)
\end{aligned}$$

   At this point we must investigate the behavior of $a^3\ln a$ as $a \to 0$, so

$$
\begin{aligned}
\lim_{a\to 0^+} a^3 \ln a &= \lim_{a\to 0^+} \frac{\ln a}{a^{-3}} \\
&= \lim_{a\to 0^+} \frac{1/a}{-3a^{-4}} \\
&= \lim_{a\to 0^+} \frac{-1}{3} a^3 = 0.
\end{aligned}
$$

Hence the integral is equal to

$$\frac{8}{3}\ln 2 - \frac{8}{9}$$

and is therefore CONVERGENT.

2. Use the Comparison Theorem to determine whether the integral is convergent or divergent. (Hint: One is convergent and one is divergent.)

   a. $\int_1^\infty \frac{1}{x+e^{2x}}dx$

      First we note that for $x \geq 1$, $x + e^{2x} \geq e^{2x}$, so that in particular we have

$$0 \leq \frac{1}{x+e^{2x}} \leq \frac{1}{e^{2x}}.$$

$$
\begin{aligned}
\int_1^\infty \frac{1}{e^{2x}}dx &= \lim_{A\to\infty} \int_1^A e^{-2x}dx \\
&= \lim_{A\to\infty} -\frac{1}{2}e^{-2x}|_1^A \\
&= \lim_{A\to\infty} -\frac{1}{2}(e^{-2A} - e^{-2}) \\
&= e^{-2}/2 < \infty.
\end{aligned}
$$

      Therefore, by the Comparison Theorem, the integral is CONVERGENT.

   b. $\int_0^{\pi/2} \frac{1}{x\sin x}dx$ Again, when in doubt start with the simple expression $x\sin x$. For $0 < x \leq \pi/2$, we have $0 < x\sin x \leq x$, and so

$$0 \leq \frac{1}{x} \leq \frac{1}{x\sin x}$$

      We then compute

$$
\begin{aligned}
\int_0^{\pi/2} \frac{1}{x}dx &= \lim_{A\to 0^+} \int_A^{\pi/2} \frac{1}{x}dx \\
&= \lim_{A\to 0^+} \ln(\pi/2) - \ln(A) \\
&= \infty,
\end{aligned}
$$

      since $\lim_{A\to 0^+} \ln A = -\infty$. Therefore, by the Comparison Theorem, the integral is DIVERGENT.