1. Determine whether each integral is convergent or divergent. Evaluate those that are convergent.

a. $\int_1^\infty \frac{1}{(3x+1)^2} \, dx$

   There is a point of discontinuity at $x = 1/3$, but this is not in our region of integration, so we do not need to split up the integral.

   \[
   \int_1^\infty \frac{1}{(3x+1)^2} \, dx = \lim_{a \to \infty} \int_1^a \frac{1}{(3x+1)^2} \, dx \\
   u = 3x + 1 \quad , \quad du = 3 \, dx \\
   = \lim_{a \to \infty} \frac{1}{3} \int_1^a u^{-2} \, du \\
   = \lim_{a \to \infty} \frac{1}{3} \left(-u^{-1}\right)_1^a \\
   = \lim_{a \to \infty} \frac{1}{3} \left(1 - \frac{1}{a}\right) \\
   = \frac{1}{12} \quad < \infty
   \]

   Therefore the integral is CONVERGENT.

b. $\int_0^{2\pi} \sin \theta \, d\theta$

   \[
   \int_0^{2\pi} \sin \theta \, d\theta = \lim_{a \to \infty} \int_0^{a} \sin \theta \, d\theta \\
   = \lim_{a \to \infty} -\cos \theta|_0^{a} \\
   = \lim_{a \to \infty} \cos 2\pi - \cos a \\
   = \lim_{a \to \infty} 1 - \cos a,
   \]

   but $\lim_{a \to \infty} \cos a$ does not exist, hence the integral is DIVERGENT.

c. $\int_0^2 z^2 \ln z \, dz$

   \[
   \int_0^2 z^2 \ln z \, dz = \lim_{a \to 0^+} \int_a^2 z^2 \ln z \, dz \\
   u = \ln z \quad , \quad dv = z^2 \, dz \\
   du = 1/z \, dz \quad , \quad v = z^3/3 \\
   = \lim_{a \to 0^+} \frac{1}{3} z^3 \ln z|_a^2 - \int_a^2 \frac{1}{3} z^2 \, dz \\
   = \lim_{a \to 0^+} \frac{1}{3} \left(8 \ln 2 - a^3 \ln a\right) - \frac{1}{9} \left(8 - a^3\right)
   \]

   At this point we must investigate the behavior of $a^3 \ln a$ as $a \to 0$, so
\[
\lim_{a \to 0^+} a^3 \ln a = \lim_{a \to 0^+} \frac{\ln a}{a^3} \\
= \lim_{a \to 0^+} \frac{1}{a} - \frac{3}{a^3} \\
= \lim_{a \to 0^+} -\frac{1}{3} a^3 = 0.
\]

Hence the integral is equal to

\[
\frac{8}{3} \ln 2 - \frac{8}{9}
\]

and is therefore CONVERGENT.

2. Use the Comparison Theorem to determine whether the integral is convergent or divergent. (Hint: One is convergent and one is divergent.)

\textbf{a.} \int_1^\infty \frac{1}{x + e^{2x}} \, dx

First we note that for \( x \geq 1 \), \( x + e^{2x} \geq e^{2x} \), so that in particular we have

\[
0 \leq \frac{1}{x + e^{2x}} \leq \frac{1}{e^{2x}}.
\]

\[
\int_1^\infty \frac{1}{e^{2x}} \, dx = \lim_{A \to \infty} \int_1^A e^{-2x} \, dx
\]

\[
= \lim_{A \to \infty} -\frac{1}{2} e^{-2x} \bigg|_1^A
\]

\[
= \lim_{A \to \infty} -\frac{1}{2} (e^{-2A} - e^{-2})
\]

\[
= e^{-2}/2 < \infty.
\]

Therefore, by the Comparison Theorem, the integral is CONVERGENT.

\textbf{b.} \int_0^{\pi/2} \frac{1}{x \sin x} \, dx

Again, when in doubt start with the simple expression \( x \sin x \). For \( 0 < x \leq \pi/2 \), we have \( 0 < x \sin x \leq x \), and so

\[
0 \leq \frac{1}{x} \leq \frac{1}{x \sin x}
\]

We then compute

\[
\int_0^{\pi/2} \frac{1}{x} \, dx = \lim_{A \to 0^+} \int_0^{\pi/2} \frac{1}{x} \, dx
\]

\[
= \lim_{A \to 0^+} \ln(\pi/2) - \ln(A)
\]

\[
= \infty,
\]

since \( \lim_{A \to 0^+} \ln A = -\infty \). Therefore, by the Comparison Theorem, the integral is DIVERGENT.