MATH 126 SPRING 2011, QUIZ 3

1. Find the limit

   a. $\lim_{x\to\infty} x\tan(1/x)$

$$\begin{aligned}\lim_{x\to\infty} x\tan(1/x) &= \lim_{x\to\infty}\frac{\tan(1/x)}{1/x}\\ &= \lim_{x\to\infty}\frac{\sec^2(1/x)(-1/x^2)}{-1/x^2}\\ &= \lim_{x\to\infty}\frac{1}{\cos^2(1/x)}\\ &= \frac{1}{\cos^2(0)}\\ &= 1.\end{aligned}$$

   b. $\lim_{x\to 0^+}(\cos x)^{1/x^2}$

   We will compute the ln of the limit first and then exponentiate

$$\begin{aligned}\lim_{x\to 0^+}\ln(\cos x)^{1/x^2} &= \lim_{x\to 0^+}\frac{1}{x^2}\ln(\cos x)\\ &= \lim_{x\to 0^+}\frac{\ln(\cos x)}{x^2}\\ &= \lim_{x\to 0^+}\frac{-\sin x/\cos x}{2x}.\end{aligned}$$

   At this point there are several techniques that will work, including using L'Hospital's rule again. Instead we will use a trick from Math 125,

$$\begin{aligned}&= \lim_{x\to 0^+}\underbrace{\left(\frac{\sin x}{x}\right)}_{\to 1}\underbrace{\left(\frac{-\cos x}{2}\right)}_{\to -1/2}\\ &= 1\cdot\frac{-1}{2} = -\frac{1}{2}.\end{aligned}$$

   Our final answer is thus

$$e^{-1/2}.$$

2. Evaluate the integral

a. $\int x^5 \ln(x)\,dx$

$$u = \ln x \qquad dv = x^5 dx$$

$$du = 1/x\,dx \qquad v = x^6/6$$

$$\begin{aligned} &= \frac{x^6}{6}\ln x - \int x^5/6dx \\ &= \frac{x^6}{6}\ln x - x^6/36 + C. \end{aligned}$$

b. $\int_1^4 e^{\sqrt{x}}\,dx$

$$\begin{aligned} \int_1^4 e^{\sqrt{x}}\,dx &= \int_1^4 e^{\sqrt{x}}\frac{2\sqrt{x}}{2\sqrt{x}}\,dx \\ t = \sqrt{x} & \qquad dt = \frac{dx}{2\sqrt{x}} \\ &= \int_1^2 e^t 2t dt \\ &= 2\int_1^2 te^t dt \\ u = t & \qquad dv = e^t dt \\ du = dt & \qquad v = e^t \\ &= 2\left(te^t|_1^2 - \int_1^2 e^t dt\right) \\ &= 4e^2 - 2e - 2e^t|_1^2 \\ &= 4e^2 - 2e - 2e^2 + 2e \\ &= 2e^2. \end{aligned}$$

c. $\int_0^{1/2} \sin^{-1}(x)\,dx$.

$$
\begin{aligned}
u = \sin^{-1}(x) & \qquad dv = dx \\
du = \frac{1}{\sqrt{1-x^2}} & \qquad v = x \\
\int_0^{1/2} \sin^{-1}(x)\,dx &= x\sin^{-1}(x)|_0^{1/2} - \int_0^{1/2} \frac{x}{\sqrt{1-x^2}}dx \\
u = 1 - x^2 & \qquad du = -2xdx \\
&= \sin^{-1}(-1/2)/2 + \frac{1}{2}\int_1^{3/4} \frac{1}{\sqrt{u}}du \\
&= \frac{\pi}{12} + \frac{1}{2}u^{1/2}/(1/2)|_1^{3/4} \\
&= \frac{\pi}{12} + \sqrt{3/4} - 1 \\
&= \frac{\pi}{12} - 1 + \sqrt{3}/2
\end{aligned}
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