MATH 126 SPRING 2011, QUIZ 12

You may assume that $\ln(1+x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1}$, for $|x| < 1$.

(1) Find the Maclaurin Series for $f(x) = \frac{\ln(1+x^4)}{x^2}$.

$$f(x) = \frac{\ln(1+x^4)}{x^2} = \sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+2}}{n+1}, \qquad |x| < 1.$$

The $|x| < 1$ comes from the fact that $|x^4| < 1 \iff |x| < 1$.

(2) What is $f^{(18)}(0)$?

We get an x^{18} term when $n = 4$, so by equating coefficients we have

$$\frac{1}{5} = \frac{f^{(18)}(0)}{18!},$$

or in other words

$$f^{(18)}(0) = \frac{18!}{5}.$$

(3) Evaluate $\int_0^{0.5} f(x)dx$ as a power series.

We integrate term by term to get

$$\begin{aligned}
\int_0^{0.5} f(x)dx &= \int_0^{0.5} \sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+2}}{n+1}\, dx \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \int_0^{0.5} x^{4n+2}\, dx \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \frac{x^{4n+3}}{4n+3}\Big|_0^{0.5} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \frac{(1/2)^{4n+3}}{4n+3}.
\end{aligned}$$

(4) Approximate $g(x) = x^{1/3}$ by its Taylor polynomial of degree 3 at $a = 1$.

The first few derivatives of $g(x)$ are

n	$g^{(n)}(x)$	$g^{(n)}(1)$
0	$x^{1/3}$	1
1	$(1/3)x^{-2/3}$	$1/3$
2	$(-2/9)x^{-5/3}$	$-2/9$
3	$(10/27)x^{-8/3}$	$10/27.$

The answer is thus

$$T_3(x) = 1 + \frac{1}{3}(x-1) - \frac{2}{18}(x-1)^2 + \frac{10}{27*6}(x-1)^3.$$

(5) Use Taylor's formula to estimate the accuracy of this approximation (from part (4)) at $x = 0.5$ (it is not necessary to simplify your answer). We will take one more derivative, so $g^{(4)}(x) = -80/81x^{-11/3}$. We have

$$|error| = |R_3(z)| = \left| \frac{-80/81z^{-11/3}}{4!}(x-1)^4 \right|,$$

where $x = 0.5 \leq z \leq 1 = a$. Since we are trying to find an upper bound, we will take z as *small* as possible, so we'll let $z = 0.5$, and we obtain

$$|error| \leq \left| \frac{-80/81}{(0.5)^{11/3} * 24}(0.5)^4 \right|.$$

Since this is just a number, we are done.