# $\epsilon - \delta$ PROBLEMS

STEPHEN A DESALVO

This document presents a 3-step procedure for finding $\delta$ when $\epsilon$ is given and the function $f(x)$ is *monotonic* (that is, is either increasing or decreasing) in the region of interest.

1. Compute $x_1 = f^{-1}(L+\epsilon)$, $x_2 = f^{-1}(L-\epsilon)$.
2. Let $\delta_1 = |a - x_1|$, $\delta_2 = |a - x_2|$.
3. Set $\delta = \min(\delta_1, \delta_2)$.

Note that $f^{-1}(x)$ represents the inverse function of $f(x)$. So for instance, if $f(x) = x^4$, then $f^{-1}(x) = \sqrt[4]{x}$.

Take now, for example, Problem 23 from Section 1.3. Here we have $f(x) = 1/x$, $f^{-1}(x) = 1/x$, $\epsilon = 0.2$, $L = 0.5$. Then we get:

1. $x_1 = f^{-1}(.7) = 10/7$, $x_2 = f^{-1}(.3) = 10/3$.
2. $\delta_1 = |2 - 10/7| = 4/7$, $\delta_2 = |2 - 10/3| = 4/3$.
3. $\delta = \min(4/7, 4/3) = 4/7$.

Therefore the answer is $\delta = 4/7$.

The reason why the third step is necessary is because the numbers $\delta_1$ and $\delta_2$ serve to restrict the deviation from $a$ (the point that $x$ is tending toward) in the left and right directions. What we are after in particular is a *single* value that works on both sides, so as long as we take the smaller one we are safe.

Further Questions for discussion:

Question: Will this procedure always work?
Ans: No! Be careful with your function. If it isn't monotonic in the region of interest then this procedure may lead you astray.
Question: Can this work with, say, $\sin(x)$?
Ans: It can! But be very careful about the values of $L+\epsilon$ and $L-\epsilon$. The above procedure worked because the function could be mapped back into a region in $x$ where $f(x)$ was monotonic. $\sin(x)$ is monotonic for small enough $\epsilon$ at every point except the points where it achieves a minimum or maximum.
Question: Can this be simplified into a very ugly 1-step procedure?
Ans: Yes:

$$\delta = \min(|a - f^{-1}(L+\epsilon)|, |a - f^{-1}(L-\epsilon)|) \tag{1}$$

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