This document is designed to clarify the concept and domain of a composition. A composition is simply a way to run an input through two or more functions to get an output. For instance, a function is like a water purifier. You input any kind of water, and it returns also water but with slightly different properties. Once you have this filtered water, you can then run it through a coffee filter, and the output is a brown liquid with a peculiar taste.

The example above illustrates a composition of two functions

\[
\text{water} \xrightarrow{\text{water filter}} \text{coffee filter} \xrightarrow{\text{coffee}}
\]

For a concrete math example, let’s have \( f(x) = \sqrt{2x + 3} \) and \( g(x) = x^2 + 1 \). If we take the composition of \( g \) with itself, we get

\[
(g \circ g)(x) = (x^2 + 1)^2 + 1 = x^4 + 2x^2 + 2,
\]

which has domain \( \mathbb{R} \) since it is just a polynomial. However, the composition above is different from the function \( k(x) = x^4 + 2x^2 + 2 \), which takes an input \( x \) and outputs according to the single formula given in \( k(x) \).

To see the distinction, let us consider

\[
(g \circ f)(x) = g(f(x)) = g(\sqrt{2x + 3}) = (\sqrt{2x + 3})^2 + 1.
\]

It is tempting to simplify this to \( 2x + 4 \) and write that the domain is all real numbers. But we must be careful! Let us think more closely about the difference between \( (g \circ f)(x) \) and \( h(x) = 2x + 4 \). \( h(x) \) is a single input output function that works for any real number because it is a polynomial. The composition, on the other hand, first must run through \( f(x) \) and then go through \( g(x) \), so the domain of \( (g \circ f)(x) \) cannot include points that are not in the domain of \( f(x) \).

With this in mind, we have

- the domain of \( h(x) \) is all real numbers, \( \mathbb{R} \), and
- the domain of \( (g \circ f)(x) \) is \( x \geq -3/2 \),

since the domain of \( f(x) \) is \( x \geq -3/2 \). Of course, one must also make sure that the range of \( f(x) \) is in the domain of \( g(x) \). This concept should already be familiar with the reader.

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