Let $N(t)$ be the number of arrivals in $[0,t]$. 
(a) $N_1(t)$ be the number of arrivals in $[0,t]$ in line 1 
    $N_2(t)$ be the number of arrivals in $[0,t]$ in line 2.

Note that $N(t) = N_1(t) + N_2(t)$.

\[ P(N_1(t) = n, N_2(t) = m) = P(N_1(t) = n, N_2(t) = m, N(t) = n+m) \]
\[ = P(N(t) = n+m) \cdot P(N_1(t) = n, N_2(t) = m | N(t) = n+m) \]

** $N(t)$ is a Poisson process with rate $\lambda$

** $P(N(t) = n+m) = \frac{(\lambda t)^{n+m}}{(n+m)!} e^{-\lambda t}$

Define $X_i = \begin{cases} 1 & \text{if the } i\text{th arrival is routed to line 1} \\ 0 & \text{if the } i\text{th arrival is routed to line 2} \end{cases}$

Bernoulli R.V.

\[ \Rightarrow X_i = \begin{cases} 1 & \text{with prob. } p \\ 0 & \text{with prob. } 1-p \end{cases} \]

\[ P(N_1(t) = n, N_2(t) = m | N(t) = n+m) = P\left( \sum_{i=1}^{n+m} X_i = n \right) \]

** Each packet is routed independently

** $X_1, X_2, X_3, \ldots$ are independent R.V.'s

** $\sum_{i=1}^{n+m} X_i$ has a binomial distribution

\[ P(\sum_{i=1}^{n+m} X_i = m) = \binom{n+m}{n} p^n (1-p)^m = \frac{(n+m)!}{n! \cdot m!} \cdot p^n (1-p)^m \]

Therefore
\[ P(N_1(t) = n, N_2(t) = m) = \frac{(\lambda t)^{n+m}}{(n+m)!} e^{-\lambda t} \cdot \frac{(n+m)!}{n! \cdot m!} \cdot p^n (1-p)^m \]

\[ = \frac{(\lambda t)^n e^{-\lambda t}}{n!} \cdot \frac{m^l}{m!} \cdot \frac{(1-p)^m e^{-(1-p)\lambda t}}{m^l} \]

$P(N_1(t) = n) = \sum_{m=0}^{\infty} P(N_1(t) = n, N_2(t) = m) = \frac{(\lambda t)^n e^{-\lambda t}}{n!} \cdot \frac{1}{m!} \cdot \frac{(1-p)^m e^{-(1-p)\lambda t}}{m^l}$ \rightarrow \text{Poisson (rate }= p\lambda\text{)}

$P(N_2(t) = m) = \sum_{n=0}^{\infty} P(N_1(t) = n, N_2(t) = m) = \frac{(\lambda t)^n e^{-\lambda t}}{n!} \cdot \frac{1}{m!} \cdot \frac{(1-p)^m e^{-(1-p)\lambda t}}{m^l}$ \rightarrow \text{Poisson (rate }= (1-p)\lambda\text{)}

$N_1(t)$ and $N_2(t)$ are independent.
Upon arrival of a customer, the probability of $n$ customers in the system is $P_n = (1 - \frac{\lambda}{\mu})^n (\frac{\lambda}{\mu})^n$.

$n$ customers in the system \[ \xrightarrow{\text{in service}} n - 1 \text{ in the queue} \]

\[ (n > 1) \]

\[ \xrightarrow{\text{in service}} \]

Let $T_1, T_2, \ldots, T_{n-1}$ be the service time of the customers that were in the queue when the new customer arrived.

\[ T_i : f_{T_i}(t) = \mu e^{-\mu t} \quad t \geq 0 \]

Let $T_0$ be the residual service time of the customer that was in service when the new customer arrived.

\[ T_0 : f_{T_0}(t) = \mu e^{\mu t} \quad t \geq 0 \] because of the memoryless property of the exponential distribution.

Let $T_w$ be the waiting time of the newly arriving customer.

Given that there are $n$ customers in the system

\[ T_w = \sum_{i=0}^{n-1} T_i \]

$T_1, T_2, \ldots, T_{n-1}$ are independent and identically distributed.

\[ \left( \sum_{i=0}^{n-1} T_i \right) \text{ follows a Gamma distribution } (\alpha = n, \beta = \frac{1}{\mu}) \]

pdf: \[ \frac{x^{n-1} e^{-x/\mu}}{(n-1)!} \]

Note: \[ \mu e^{-x/\mu} \leftarrow (1 - j\omega \frac{1}{\mu})^{-1} \]

\[ x^{\alpha-1} e^{-x/\beta} \leftarrow (1 - j\omega \beta)^{-\alpha} \]

\[ \Gamma(n+1) = n! \]

Note that if there are no customers "upon" arrival, the waiting time is 0 (deterministic).

The system time seen by the newly arriving customer is

\[ T = T_w + T_0 \]

service time of the newly arriving customer

\[ T_0 : f_{T_0}(t) = \mu e^{\mu t} \quad t \geq 0 \]

Given that there are $n$ customers in the system

\[ T \sim \text{Gamma}(\alpha = n + 1, \beta = \frac{1}{\mu}) \]

pdf: \[ \frac{x^{n+1-1} e^{-x/\mu}}{(n+1)!} \]
Let $N$ be the number of customers in the system (right before the new arrival)

$$f_{Tw}(t) = \sum_{n=0}^{\infty} f_{Tw|N}(t|n) \ P\{N=n\} \quad (\text{total probability})$$

$$P\{N=n\} = p_n = (1-\frac{\lambda}{\mu})(\frac{\lambda}{\mu})^n$$

$$f_{Tw|N}(t|n) = \begin{cases} s(t) & n=0 \\ \frac{p^nt^{n-1}e^{-\mu t}}{(n-1)!} & n>1 \end{cases}$$

$$f_{Tw}(t) = (1-\frac{\lambda}{\mu})s(t) + \sum_{n=1}^{\infty} \frac{p^nt^{n-1}e^{-\mu t}}{(n-1)!} \ (1-\frac{\lambda}{\mu})(\frac{\lambda}{\mu})^n$$

$$= (1-\frac{\lambda}{\mu})s(t) + (1-\frac{\lambda}{\mu})e^{\mu t} \sum_{n=1}^{\infty} \frac{(\lambda t)^{n-1}}{(n-1)!}$$

$$\therefore f_{Tw}(t) = (1-\frac{\lambda}{\mu})s(t) + (\frac{\lambda}{\mu})(\mu-\lambda) e^{(\mu-\lambda)t} \quad t \geq 0$$

$$f_T(t) = \sum_{n=0}^{\infty} f_{T|N}(t|n) \ P\{N=n\}$$

$$f_{T|N}(t|n) = \frac{p^n t^n e^{-\mu t}}{n!} \quad n=0,1,2,\ldots$$

$$f_T(t) = \sum_{n=0}^{\infty} \frac{p^n t^n e^{-\mu t}}{n!} \ (1-\frac{\lambda}{\mu})(\frac{\lambda}{\mu})^n$$

$$= (\mu-\lambda) e^{-\mu t} \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!}$$

$$\therefore f_T(t) = (\mu-\lambda) e^{-(\mu-\lambda)t} \quad t \geq 0$$