Linear Prediction Based Semi-Blind Channel Estimation for MIMO-OFDM System

Feng Wan, Wei-Ping Zhu and M.N.S. Swamy
Centre for Signal Processing and Communications
Dept. of Electrical and Computer Engineering, Concordia University
Montreal, Quebec, Canada H3G 1M8

***Abstract*— In this paper, a semi-blind channel estimation method is presented for MIMO-OFDM systems. The new method uses the linear prediction for obtaining a blind constraint on the MIMO-OFDM channel matrix as well as the least-squares approximation for the training signal. The proposed method can be regarded as an extension of an existing semi-blind MIMO channel estimation algorithm. Yet the extension is nontrivial, since the formulation of the MIMO-OFDM signal and the related blind constraint cannot easily be obtained from the MIMO counterpart. The proposed algorithm is simulated using Monte-Carlo method and compared with the LS method in terms of the mean square error (MSE) of the estimation. Simulation results show that the proposed method consistently outperforms the LS method when the same training signal is used.**

I. Introduction

MIMO-OFDM (Multiple input multiple output-orthogonal frequency division multiplexing) technology, a new research trend in wireless communications, has been considered as a strong candidate for future wireless communication systems. With multiple transmit and multiple receive antennas, it can achieve a high data rate without increasing the total transmission power or bandwidth, compared with single antenna systems. Furthermore, its frequency-selective problem has been solved by OFDM technique. It has been proved that when perfect knowledge of the wireless channel is available at the receiver, the capacity of MIMO-OFDM systems grows linearly with the number of antennas. In practice, however, the channel condition is not known. Therefore, channel estimation, also known as channel identification, plays a key role in MIMO-OFDM systems.

Broadly speaking, there are three kinds of channel estimation approaches. First, the training-based methods [1], for example, least square (LS), maximum likelihood (ML), maximum a posteriori (MAP) and maximum mean square error (MMSE) algorithms, employ known training signals to render an accurate channel estimation. In contrast to training-based methods, blind channel estimation algorithms, such as those proposed in [2]–[4], which exploit the second-order cyclostationary statistics, correlative coding and other properties, have a better spectral efficiency. With the idea of combing the advantages of both the training-based and the blind algorithms, semi-blind channel estimation techniques, which can potentially enhance the quality of MIMO channel estimation, have been proposed [3], [5], [6]. With a small number of training symbols, problems such as ambiguities and mis-convergence of the blind methods can be solved. On the other hand, the use of the available data in semi-blind techniques can improve the accuracy of channel estimation.

In this paper, we focus on a class of semi-blind channel estimation methods based on MIMO linear prediction technique. Linear prediction has been widely used in blind MIMO channel estimation and equalization [2], [5], [7], [8]. The key idea in this technique is to represent the received MIMO signal as a finite-order *autoregressive* (AR) series providing that the transmitted signal is uncorrelated in time [2]. Based on the AR series, a linear prediction filter can be obtained and used for second-order deconvolution. By combining the linear prediction with a higher-order statistics (HOS) or the weighted least-squares method, some blind channel estimation algorithms have been derived [2]. However, these algorithms require a large sample size and are not robust. Alternatively, a semi-blind algorithm that uses the linear prediction as the blind constraint, in combination with some training data, have been proposed [3], [5], [6], yielding a better estimation performance. However, this semi-blind estimation algorithm is not applicable to MIMO-OFDM systems, since the signal model in MIMO-OFDM systems is quite different from that in MIMO systems. In this paper, we will extend this approach for the estimation of MIMO-OFDM channels, in which we give a new correlation matrix estimation method by using the circular convolution property.

Throughout the paper, we adopt the following notations:
$\dagger$ Pseudo-inverse,
$\otimes$ Kronecker product,
T Transpose,
H Complex conjugate transpose,
$\| \|_F$ Frobenius norm, and
$\mathrm{vec}\,()$ a stacking of the columns of the involved matrix into a vector.

II. Brief Overview of Linear Prediction for MIMO Channel Estimation

Consider an MIMO system with N_T transmit and N_R receive antennas. Assume that the channel is an array of L-tap FIR filters characterized by a number of $N_R \times N_T$ matrices $\mathbf{H}\,(n)\,(n = 0, 1, \cdots, L-1)$, whose (i_R, i_T)-th element $h_{i_T,i_R}\,(n)$ represents the channel response from the i_T-th transmit antenna to the i_R-th receive antenna. For the transmitted signal vector $\mathbf{x}\,(n) \triangleq [x_1\,(n), \cdots, x_{N_T}\,(n)]^T$

consisting of uncorrelated signals, the received signal vector can be written as $\mathbf{y}(n) \triangleq [y_1(n), \cdots, y_{N_R}(n)]^T$, with its element being given by

$$y_{i_R}(n) = \sum_{i_T=1}^{N_T} h_{i_R,i_T}(n) * x_{i_T}(n) + v_{i_R}(n) \quad (1)$$

where $*$ denotes the linear convolution and $v_{i_R}(n)$ is a spatio-temporally uncorrelated noise with variance δ_v^2.

We would first like to briefly review the MIMO linear prediction and the related semi-blind channel estimation method [2]. Define

$$\begin{aligned} \mathbf{y}_P(n-1) &\triangleq \left[\mathbf{y}^T(n-1), \cdots, \mathbf{y}^T(n-P)\right]^T, \\ \tilde{\mathbf{R}}_{n-1} &\triangleq \mathrm{E}\left\{\mathbf{y}_P(n-1)\mathbf{y}_P^H(n-1)\right\}, \quad (2) \\ \ddot{\mathbf{R}}_n &\triangleq \mathrm{E}\left\{\mathbf{y}(n)\mathbf{y}_P^H(n-1)\right\}. \end{aligned}$$

The MIMO linear predictor can be written as

$$\mathbf{P}_P \triangleq [\mathbf{P}_P(1), \mathbf{P}_P(2), \cdots, \mathbf{P}_P(P)] = \ddot{\mathbf{R}}_n \tilde{\mathbf{R}}_{n-1}^{-1} \quad (3)$$

where $\mathbf{P}_P(n)$, $(n = 1, \cdots, P)$ is an $N_R \times N_R$ matrix representing the n-th tap of the prediction filter. The covariance matrix of the prediction error can be then given by

$$\boldsymbol{\delta}_{\tilde{\mathbf{y}},P}^2 = \mathbf{R}(0) - \mathbf{P}_P \ddot{\mathbf{R}}_n^H \quad (4)$$

where $\mathbf{R}(0) \triangleq \mathrm{E}\left[\mathbf{y}(n)\mathbf{y}^H(n)\right]$. Further, define $\mathbf{P}_P(z) = \mathbf{I} - \sum_{i=1}^{P} \mathbf{P}_P(i) z^{-i}$, $\mathbf{H}(z) = \sum_{i=0}^{L-1} \mathbf{H}(i) z^{-i}$. It has been shown in [2] that if $PN_R \geq (L + P - 1) N_T$, one can obtain

$$\mathbf{P}_P(z)\mathbf{H}(z) = \mathbf{H}(0), \quad (5)$$

$$\boldsymbol{\delta}_{\tilde{\mathbf{y}},P}^2 = \mathbf{H}(0)\mathbf{H}^H(0). \quad (6)$$

Based on (5) and (6), some blind channel estimation algorithms have been proposed [2], [9]. They first acquire an estimate of $\mathbf{H}(0)$ from the estimate of $\boldsymbol{\delta}_{\tilde{\mathbf{y}},P}^2$, and then use it to estimate the channel matrix $\mathbf{H}(z)$.

By using the above MIMO linear prediction method along with training data, a semi-blind approach for MIMO channel estimation has been developed to achieve a better estimation performance [3], [5], [6]. The basic idea of this approach is briefly described as follows.

Denote the null column space of $\mathbf{H}(0)$ as $\mathbf{U}_{0null}$. From (6), one can easily find that $\mathbf{U}_{0null}$ can be estimated from $\boldsymbol{\delta}_{\tilde{\mathbf{y}},P}^2$. By using $\mathbf{U}_{0null}$ into (5), we have

$$\mathbf{U}_{0null}^H \mathbf{H}(0) = \mathbf{U}_{0null}^H \mathbf{P}_P(z)\mathbf{H}(z) = \mathbf{0}, \quad (7)$$

implying that $\mathbf{U}_{0null}^H \mathbf{P}_P(z)$ is a blind constraint on the channel matrix. This constraint can then be used to derive a blind constraint $\bar{\mathbf{B}}$ for the channel vector $\mathbf{h}_s$, namely,

$$\mathbf{h}_s \triangleq \left[\mathbf{h}_{1,s}^T(0), \cdots, \mathbf{h}_{N_T,s}^T(0), \cdots, \mathbf{h}_{N_T,s}^T(L-1)\right]^T$$

with $\mathbf{h}_{i_T,s}(l) = [h_{1,i_T}(l), \cdots, h_{N_R,i_T}(l)]^T$. By combining the blind constraint with a training-based least square (LS) criterion [5], [6], the semi-blind algorithm can be formulated as

$$\min_{\hat{\mathbf{h}}_s} \left\{ \left\| \mathbf{Y}_{\mathrm{TS}} - \mathbf{A}_{\mathrm{TS}} \hat{\mathbf{h}}_s \right\|_F^2 + \alpha \left\| \bar{\mathbf{B}} \hat{\mathbf{h}}_s \right\|_F^2 \right\} \quad (8)$$

where $\mathbf{A}_{\mathrm{TS}}$ is a pilot signal matrix, $\mathbf{Y}_{\mathrm{TS}}$ the corresponding received signal vector, and $\alpha > 0$ is a weighting factor.

III. Proposed MIMO-OFDM Semi-Blind Channel Estimation Algorithm

For the sake of simplicity, only one OFDM symbol with K subcarriers is considered. Let $\mathbf{x}(n)$ be the transmitted time-domain signal vector at the output of the IDFT module, and $\mathbf{y}(n)$ be the received time-domain signal vector before the DFT operation. Then, the i_R-th received signal can be written as

$$y_{i_R}(n) = \sum_{i_T=1}^{N_T} h_{i_R,i_T}(n) \circledast x_{i_T}(n) + v_{i_R}(n). \quad (9)$$

Note that unlike the MIMO signal model in (1), the MIMO-OFDM signal given by (9) involves circular convolution. In what follows, we will develop a semi-blind MIMO-OFDM channel estimation algorithm based on (9) as an extension of the semi-blind criterion in (8).

A. Training based LS Criterion

We first consider the LS criterion for the training signal. In MIMO-OFDM systems, the training signal is transmitted in the frequency domain and thus, the LS part in (8) should be modified. Following the approach in [10], the received signal at the i_R-th receive antenna can be obtained as

$$\mathbf{Y}_{i_R,\mathrm{pilot}} = \mathbf{A}\mathbf{h}_{i_R} + \boldsymbol{\xi}_{i_R,\mathrm{pilot}} \quad (10)$$

where $\boldsymbol{\xi}_{i_R,\mathrm{pilot}}$ is the noise vector with respect to the i_R-th receive antenna, $\mathbf{A}$ is a matrix carrying on the pilot signal, and $\mathbf{h}_{i_R}$ the one-path channel response vector, as given by

$$\mathbf{h}_{i_R} \triangleq [h_{n_R,1}(0), \cdots, h_{n_R,1}(L-1), \cdots, h_{n_R,n_T}(L-1)]^T. \quad (11)$$

By defining

$$\begin{aligned} \bar{\mathbf{Y}}_{\mathrm{pilot}} &\triangleq [\mathbf{Y}_{1,\mathrm{pilot}}, \cdots, \mathbf{Y}_{N_R,\mathrm{pilot}}], \\ \mathbf{H} &\triangleq [\mathbf{h}_1, \cdots, \mathbf{h}_{N_R}], \quad (12) \\ \bar{\boldsymbol{\xi}}_{\mathrm{pilot}} &\triangleq [\boldsymbol{\xi}_{1,\mathrm{pilot}}, \cdots, \boldsymbol{\xi}_{N_R,\mathrm{pilot}}], \end{aligned}$$

we can obtain:

$$\bar{\mathbf{Y}}_{\mathrm{pilot}} = \mathbf{A}\mathbf{H} + \bar{\boldsymbol{\xi}}_{\mathrm{pilot}}. \quad (13)$$

Further letting $\mathbf{Y}_{\mathrm{pilot}} \triangleq \mathrm{vec}\left(\bar{\mathbf{Y}}_{\mathrm{pilot}}\right)$, $\tilde{\mathbf{A}} \triangleq \mathbf{I} \otimes \mathbf{A}$ and $\mathbf{h} \triangleq \mathrm{vec}(\mathbf{H})$, from (13), one can obain a new LS criterion

$$\left\| \mathbf{Y}_{\mathrm{pilot}} - \tilde{\mathbf{A}}\hat{\mathbf{h}} \right\|_F^2, \quad (14)$$

which will be used for the training signal in the proposed semi-blind optimization.

B. Blind Criterion based on MIMO Linear Prediction

We now derive the blind constraint. Unlike MIMO systems, where the transmitted signals are considered uncorrelated, the uncorrelated signals in MIMO-OFDM systems are generated in the frequency-domain. However, we have found through a large amount of experiments that the transmitted MIMO-OFDM signals are also uncorrelated in the time-domain. As such, the MIMO linear prediction method can still be used for MIMO-OFDM systems.

By following the linear prediction process in Section II, we can obtain a time-domain representation of (5),

$$[\mathbf{I}, -\mathbf{P}_P]\,\mathbf{H}_D = [\mathbf{H}(0), \mathbf{0}, \cdots, \mathbf{0}] \tag{15}$$

where $\mathbf{P}_P$ is given by (3) and $\mathbf{H}_D$ is a $(P+1)\,N_R \times (L+P)\,N_T$ block Toeplitz matrix with the first block row as $[\mathbf{H}(0), \cdots, \mathbf{H}(L-1), \mathbf{0}, \cdots, \mathbf{0}]$. Letting $\mathbf{H}_F \triangleq \left[\mathbf{H}_0^H, \cdots, \mathbf{H}_{L-1}^H\right]^H$ and $\mathbf{P}_Q$ be an $(L+P)\,N_R \times LN_R$ block toeplitz matrix with the first block column as $\left[\mathbf{I}, -\mathbf{P}_P^H(1), \cdots, -\mathbf{P}_P^H(P), \mathbf{0}, \cdots, \mathbf{0}\right]^H$, (15) can be rewriten as

$$\mathbf{P}_Q\mathbf{H}_F = \left[\mathbf{H}_0^H, \mathbf{0}, \cdots, \mathbf{0}\right]^H. \tag{16}$$

Using (6), the null column space of $\mathbf{H}(0)$, $\mathbf{U}_{0null}$, can be easily obtained, which is then used to form

$$\mathbf{P}_\Sigma \triangleq \left(\mathbf{I}_{L+P} \otimes \mathbf{U}_{0null}^H\right)\mathbf{P}_Q. \tag{17}$$

From (16) and (17), we have

$$\mathbf{P}_\Sigma\mathbf{H}_F = \mathbf{0}, \tag{18}$$

which can be equivalently rewritten as

$$(\mathbf{I} \otimes \mathbf{P}_\Sigma)\mathrm{vec}(\mathbf{H}_F) = \mathbf{0}, \tag{19}$$

Noting that $\mathrm{vec}(\mathbf{H}_F) = \mathbf{E}_P\mathbf{h}$, where $\mathbf{E}_P$ is a known permutation matrix, (19) can be expressed as

$$(\mathbf{I} \otimes \mathbf{P}_\Sigma)\,\mathbf{E}_P\mathbf{h} = \mathbf{B}\mathbf{h} = \mathbf{0} \tag{20}$$

implying that $\mathbf{B} = (\mathbf{I} \otimes \mathbf{P}_\Sigma)\,\mathbf{E}_P$ is the blind constraint for the channel vector $\mathbf{h}$.

In the computation of the linear predictor $\mathbf{P}_P$ and the covariance matrix $\boldsymbol{\delta}^2_{\tilde{\mathbf{y}},P}$, one has to estimate various correlation matrices $\tilde{\mathbf{R}}_{n-1}$, $\ddot{\mathbf{R}}_n$ and $\mathbf{R}(0)$ as discussed in Section II. Considering that the circular convolution is used in MIMO-OFDM systems, a more accurate estimate of these correlation matrices can be obtained in comparision to that in MIMO systems. For example, the estimate of $\tilde{\mathbf{R}}_{n-1}$ in MIMO systems is computed as

$$\hat{\tilde{\mathbf{R}}}_{n-1} = \frac{1}{K}\sum_{n=P}^{K}\mathbf{y}_P(n)\,\mathbf{y}_P^H(n) - \delta_v^2\mathbf{I} \tag{21}$$

where only $K-P+1$ received signal vectors $\mathbf{y}_P(n)$ $(n = P, \cdots, K)$, are available for estimation. In MIMO-OFDM systems, however, the estimation of $\tilde{\mathbf{R}}_{n-1}$ can be modified as

$$\hat{\tilde{\mathbf{R}}}_{n-1} = \frac{1}{K}\sum_{n=1}^{K}\mathbf{y}_P(n)\,\mathbf{y}_P^H(n) - \delta_v^2\mathbf{I} \tag{22}$$

where $\mathbf{y}_P(n)$, $(n = 1, \cdots, P-1)$ can be obtained using $\mathbf{y}(n) \triangleq \mathbf{y}(K+n)$ due to the circular convolution. Since more signal samples are used in the estimate, a better linear prediction result can be expected in MIMO-OFDM systems.

C. Semi-Blind Solution

Combining (14) and (20), a semi-blind cost function for channel estimation can be formulated,

$$\min_{\hat{\mathbf{h}}}\Delta = \left\|\mathbf{Y}_{\mathrm{pilot}} - \tilde{\mathbf{A}}\hat{\mathbf{h}}\right\|_F^2 + \alpha\left\|\hat{\mathbf{B}}\hat{\mathbf{h}}\right\|_F^2 \tag{23}$$

where $\hat{\mathbf{B}}$ is an estimate of the blind constraint. The solution to this optimization problem can be obtained by letting

$$\frac{\partial\Delta}{\partial\hat{\mathbf{h}}^H} = -\tilde{\mathbf{A}}^H\left(\mathbf{Y}_{\mathrm{pilot}} - \tilde{\mathbf{A}}\hat{\mathbf{h}}\right) + \alpha\hat{\mathbf{B}}^H\hat{\mathbf{B}}\hat{\mathbf{h}} = \mathbf{0},$$

which gives

$$\hat{\mathbf{h}} = \left(\tilde{\mathbf{A}}^H\tilde{\mathbf{A}} + \alpha\hat{\mathbf{B}}^H\hat{\mathbf{B}}\right)^\dagger\tilde{\mathbf{A}}^H\mathbf{Y}_{\mathrm{pilot}}. \tag{24}$$

As the weight for a weighted least-square (WLS) minimization problem can usually be determined according to the variance of the individual estimation error [11], here we propose to calculate the weighting factor by

$$\alpha = \left(\left\|\tilde{\mathbf{A}}\right\|_F^2\mathrm{MSE_T}\right) / \left(\left\|\hat{\mathbf{B}}\right\|_F^2\mathrm{MSE_B}\right),$$

where $\mathrm{MSE_T}$ represents the MSE of the training-based LS estimation of the channel vector and $\mathrm{MSE_B}$ is the MSE of the blind criterion. It has been shown in [10] that when the optimal pilot signal is used, $\mathrm{MSE_T}$ can be calculated by $\mathrm{MSE_T} = N_RN_TL\delta_v^2/K\delta_x^2$. Although it is difficult to derive a closed-form expression for $\mathrm{MSE_B}$, it can be estimated off-line by using training signals, since we have found that $\mathrm{MSE_B}$ almost remains unchanged when the channel varies. Therefore, α can be determined prior to the calculation of $\hat{\mathbf{h}}$ as required by (24).

IV. Simulation Results

We consider an MIMO-OFDM system with 2 transmit and 4 receive antennas. The number of subcarriers is set to 512, the length of cyclic prefix is 20, and the length of the linear predictor is $P = 4$. In our simulation, the QPSK modulation is used and a Rayleigh channel modelled by a 3-tap MIMO-FIR filter is assumed, in which each tap corresponds to a 2×4 random matrix whose elements are independent identically distributed (i.i.d.) complex Gaussian variables with zero mean and unit variance.

A. Experiment 1

In the first experiment, the channel estimation performance in terms of the MSE versus the SNR is investigated. The simulation is based on 1000 Monte Carlo runs of the transmission of one OFDM symbol at 512 subcarriers of which 64 are used as pilot for training purpose. Since the blind constraint of the proposed semi-blind algorithm heavily depends on $\mathbf{H}(0)$ as seen from (6) and (7), it is necessary to investigate the effect

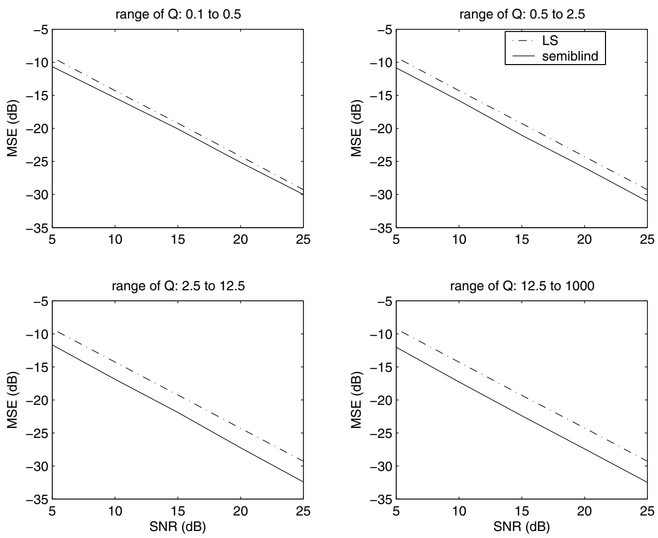

Fig. 1. MSE versus SNR for different ranges of Q value.

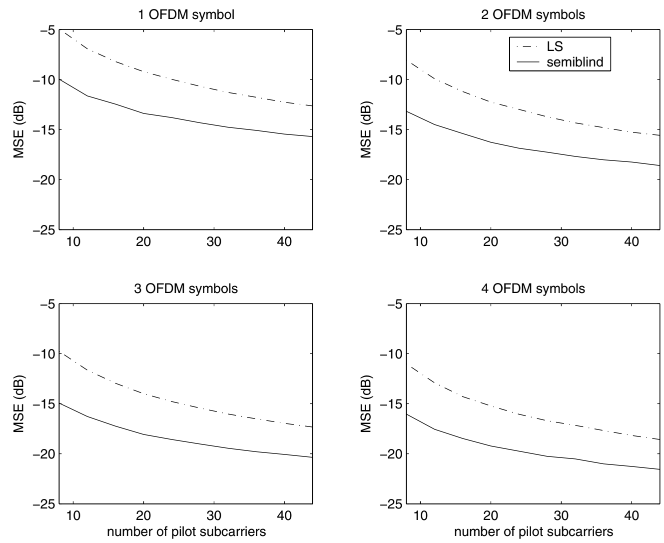

Fig. 2. MSE versus length of pilot with different number of OFDM symbols.

of $\mathbf{H}(0)$ on the channel estimation performance. To this end, we define the following metric

$$Q \triangleq \left\|\mathbf{H}(0)\,\mathbf{H}^H(0)\right\|_F^2.$$

Fig. 1 shows the MSE-SNR plots resulting from the proposed semi-blind method as well as the LS estimation for different ranges of Q. It is seen that, with the increase of the Q value, the superiority of the semi-blind algorithm over the LS algorithm becomes more prominent.

B. Experiment 2

Here we investigate the channel estimation performance of the semi-blind algorithm versus the number of OFDM symbols as well as the number of subcarriers for pilot signal, in comparison with that of the LS method. The number of OFDM symbols used in the two methods is set to be from 1 to 4, and in each of the four cases, the number of pilot subcarriers per OFDM symbol varise from 8 to 48. Fig. 2 shows the MSE plots from 500 Monte Carlo iterations for an SNR of 10 dB when $Q > 1$. It is seen that the semi-blind algorithm outperforms the LS method by 5 and 3dB when the number of pilot subcarriers is 8 and 48, respectively, implying that the proposed method is more advantageous for pilot signals with a shorter length. It is also observed that the performance improvement of the semi-blind method over the LS method is almost the same for different number of OFDM symbols used.

V. Conclusion

A semi-blind MIMO-OFDM channel estimation based on linear prediction method has been proposed. Through a proper formulation of the received signal as well as the blind constraint on the channel matrix, the new method has been shown to be able to extend an existing MIMO channel estimation algorithm for MIMO-OFDM systems. The proposed method has been simulated and compared with the training-based LS method, confirming the significant advantage of the semi-blind channel estimation approach in terms of the MSE versus different SNRs as well as different number of pilot subcarriers.

References

[1] M. Biguesh and A. B. Gershman, "MIMO channel estimation: optimal training and tradeoffs between estimation techniques," in *Proc. of IEEE International Conference on Communications*, 2004, vol. 5, pp. 2658–2662.

[2] A Gorokhov and P Loubaton, "Blind identification of MIMO-FIR systems: A generalized linear prediction approach," *Signal Processing*, vol. 73 (1-2), pp. 105–124, 1999.

[3] A. Medles and D. T. M. Slock, "Linear precoding for spatial multiplexing MIMO systems: blind channel estimation aspects," in *Proc. of IEEE International Conference on Communications*, 2002, vol. 1, pp. 401–405.

[4] Ding Zhi and Qiu Li, "Blind MIMO channel identification from second order statistics using rank deficient channel convolution matrix," *IEEE Trans. on Signal Processing*, vol. 51, no. 2, pp. 535–544, 2003.

[5] A. Medles, D. T. M. Slock, and E. De Carvalho, "Linear prediction based semi-blind estimation of MIMO FIR channels," in *Proc. of IEEE Third Workshop on Signal Processing Advances in Wireless Communications*, 2001, pp. 58–61.

[6] A. Medles and D. T. M. Slock, "Semiblind channel estimation for MIMO spatial multiplexing systems," in *Proc. of IEEE Vehicular Technology Conference*, 2001, vol. 2, pp. 1240–1244.

[7] R. Lopez-Valcarce and S. Dasgupta, "Blind channel equalization with colored sources based on second-order statistics: a linear prediction approach," *IEEE Trans. on Signal Processing*, vol. 49, no. 9, pp. 2050–2059, 2001.

[8] Y. Inouye and Liu Ruey-Wen, "A system-theoretic foundation for blind equalization of an FIR MIMO channel system," *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, vol. 49, no. 4, pp. 425–436, 2002.

[9] T.W.S. Chow, Bao-Yun Wang, and K.T. Ng, "Linear prediction based multipath channel identification algorithm," *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, vol. 50, no. 6, pp. 769 – 774, 2003.

[10] I. Barhumi, G. Leus, and M. Moonen, "Optimal training design for MIMO OFDM systems in mobile wireless channels," *IEEE Trans. on Signal Processing*, vol. 51, no. 6, pp. 1615–1624, 2003.

[11] Cesar Moreno, "A least-squares-based method for determing the ratio between two measured quantities," *Measurement Science and Technology*, vol. 7, pp. 137 – 141, 1996.