Stability of a Submerged Deformable Body

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OBJECTIVE
We study the stability of passive motion of a fish model. The (articulated body) fish model
accounts for the finite dimensions of the fish, its bending stiffness (via the torsional springs
at the joints), the inextensibility of the tissue (via the added mass effect) but it does not take
into consideration vortex shedding from the trailing edge of the fish. The stability analysis
shows that there is a range of parameter values (bending stiffness versus body dimensions) that
support stable passive-swimming in the direction of the body’s length.

SETUP
Model: Fish is modeled as a neutrally buoyant ar-
ticulated body, which moves in an infinitely large
volume of incompressible, inviscid and irratio-
nal fluid $F$ at rest at infinity. Let the body be made of $B_i$, $i = 1, \ldots, N$ identical rigid ellipses of uniform density $\rho$ with semi-axis $a$. The rigid ellipses $B_i$ are connected via $N - 1$ massless and frictionless hinge joints with torsional spring constants $k_i$. An example of three-link body is shown.

Equations of Motion: The kinetic energy $T$ of the solid-fluid system can be written as the sum of the energies of the solid links $T_B$ and the energy of the fluid $T_F$, where

$$ T = \frac{1}{2} \sum_{i} \Omega_i \cdot \dot{\Omega}_i + \frac{1}{2} \Omega_i \cdot \dot{\Omega}_i. $$

Here, $\Omega_i$ and $m\dot{\Omega}_i$ are the moment of inertia and mass of the rigid ellipsoid, respectively, and $\Omega_i$ are the linear and angular velocities of $B_i$ expressed in body-fixed frame. Follow a standard procedure, $T_F$ can be written as

$$ T_F = \frac{1}{2} \sum_{i} \Omega_i \cdot \dot{\Omega}_i + \frac{1}{2} \Omega_i \cdot \dot{\Omega}_i. $$

where $M_i$, $D_i$, $L_i$, and $M_i^2$ are referred to as the added mass matrices. For simplicity, the ellipses are assumed to be hydrodynamically decoupled, that is, the added masses for each ellipse can be computed independently of the presence of the remaining ellipses. One has that, for $i \neq j$, $M_i^2 = \text{diag}(0, 0, J_i^T)$ and $M_i^2 = \text{diag}(m_i, m_i^2, J_i^T)$ can be computed analytically. The total kinetic energy of the solid-fluid system can then be written as

$$ T = \frac{1}{2} \sum_{i} \Omega_i \cdot \dot{\Omega}_i + \frac{1}{2} \dot{\Omega}_i \cdot \Omega_i. $$

We analyze the stability of relative equilibria for body-fluid system using linearization method
then verify the findings of our linear stability analysis using numerical simulations of the non-
linear equations.

Two-link Model: Linearize (4) around the equi-
librium $q_0$, one obtains the linearized equations of motion in matrix form as

$$ \delta q = A \delta q, $$

where $\delta q$ is the linear perturbation, $A$ is a $5 \times 5$ matrix. The real part of the eigenvalues for a spe-
cific value of the non-dimensionalized aspect ratio $\theta = 0.2$ is shown as a function of spring constants $k$.
One could identify a range of $\theta$ values for which all
eigenvalues have zero real part, which implies that
the system is stable for these parameter val-
ues. A full parametric analysis of the eigenvalues when varying both $\theta$ and $k$ suggest that, indeed,
there is a region in the space $(\theta, k)$ where two-link body is linearly stable.

Three-link Model: Consider a three-link body where the links are made of identical ellipses
and 025 and 0.025 and 0.025, 0.05, 0.025, 0.052 (6) and for three sets of parameter values: $\theta = 0.2$ and $k = 0.4, 0.94$, and 1.5. The configuration of the system at $k$-th time instants is shown for all three cases.

One could verify these stability results by direct numerical integration of (4) with initial con-
tions slightly perturbed away from the relative equilibrium $q_0$. More specifically, we integrate the nonlinear equations from $t = 0$ to $t = 50$ with initial conditions $q(0) = (1, 0, 0.002, 0, 0.025)^T$, i.e. $\theta$ slightly perturbed from 0, for $k = 0.2$ (same as the linear example) and $k = 0.2, 0.4$ and 0.6. We show snapshots of the body motion at $t = 0, 50, 100, 300$, and 50 for all three cases.

Clearly, the equilibrium is stable only when $k = 0.4$, while for $k = 0.2$ the body stumbles and for $k = 0.6$ the body stumbles and intersects itself and eventually the numerical integration blows-up.

Note that, in comparison to the two-link case, the linearly stable region for $\theta < 1$ is much smaller. It is basically a portion of a curve. It is important to point out that these parametric
results are obtained by varying $\theta$ and $k$ using constant increments of 0.025 and 0.0004,
respectively. We also integrate the nonlinear equations numerically for the initial conditions $q_0 = (0.2, 0.0001, 0.01, 0.002, 0.05)^T$ and for three sets of parameter values: $\theta = 0.2$ and $k = 0.4, 0.94$, and 1.5. The configuration of the system at $k$-th time instants is shown for all three cases.

FUTURE WORKS
• Analyze the behavior of the stability region as the number of links $N$ increases and eventu-
ally as $N \rightarrow \infty$, that is, for a fully deformable body (including only bending deformations)
• Analyze stability for hydrodynamically coupled model using panel method
• Eventually couple the deformable body with vortex shedding mechanism and study the
effect of the presence of vortices

REFERENCES
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Leonard, N.E. [1997], Stability of a bottom-heavy underwater vehicle, Automatica, 33(1341-
346).

Three-link Model: Consider a three-link body where the links are made of identical ellipses
of major and minor axes $a$ and $b$ and uniform density $\rho$ and let $\theta_i$ and $\theta_i$ be the relative
angles between $B_1$, $B_2$, and $B_3$, respectively. Follow similar derivation, one obtains governing
equations in terms of $q = [x, y, \theta_1, \theta_2, \theta_3]^T$, where the velocities of $B_i$ are expressed
in the body frame: A full parametric linear stability analysis gives stability region in $(k, b)$ space,
where $k = k_1/k_2$.

$\theta_1 = 0$ and $\theta_2 = 0$.

Equation (4) around the equi-

$\delta q = A \delta q$, where $\delta q$ is the linear perturbation, $A$ is a $5 \times 5$ matrix. The real part of the eigenvalues for a spe-
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