PART I - VISCOUS EVOLUTION OF POINT VORTEX EQUILIBRIA

PART II - EFFECTS OF BODY ELASTICITY ON STABILITY OF FISH MOTION

by

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Dedication

To Yi
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0 
0.01 
0 
0 
1 
0 
0 
0 
0 
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0 
0 
0.01 
1 
0 
0 
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8.5 Numerical integration for the three-link hydrodynamically decoupled model in 3D. Initial condition is \( \mathbf{\Psi}_0 = [1 0 0 0 0 0.05 0 -0.05 0]^T \). Integration is for time from 0 to 50, with parameter values \( b = 0.2, c = 1.1 \) for all cases and \( k_1 = k_2 = 0.4 \) (top row), 0.7 (middle row) and 1 (bottom row). The left column shows snapshot sketches of the body motion at \( t = 0, 25 \) and 50 for all three cases. The middle column shows \( y \) component of trajectories of three body centers \( y_1, y_2 \) and \( y_3 \) in inertial frame as functions of time, where the horizontal axes are \( t \) in a \([0 50]\) interval, and vertical axes are \( y \) in a \([-5 5]\) interval. And the right column shows relative angles \( \theta_1 \) and \( \theta_2 \) as functions of time, horizontal scales are time in a \([0 50]\) interval, and vertical scales are \([-1 1]\). Only the \( k_1 = k_2 = 0.7 \) cases (middle row) is stable.

9.1 Summary of stability results of 2D models.

9.2 Sketch of snapshots of unstable motion for three-link hydrodynamically decoupled model in 3D. The parameters are such that in-plane and out-of-plane motions are both stable. But since the initial perturbation is in both in-plane and out-of-plane motions, the nonlinear dynamics is unstable.
Abstract

Vortex dynamics and solid-fluid interactions are two of the most important and most studied topics in fluid dynamics for their relevance to a wide range of applications from geophysical flows to locomotion in moving fluids. In this work, we investigate two problems in two parts: Part I studies the viscous evolution of point vortex equilibria; Part II studies the effects of body elasticity on the passive stability of submerged bodies.

In Part I, we describe the viscous evolution of point vortex configurations that, in the absence of viscosity, are in a state of fixed or relative equilibrium. In particular, we examine four cases, three of them correspond to relative equilibria in the inviscid point vortex model and one corresponds to a fixed equilibrium. Our goal is to elucidate the dominant transient dynamical features of the flow. A multi-Gaussian “core growing” type of model is typically used in high fidelity numerical simulations, but we propose to implement it as a low-order model for the flow field. We show that all four configurations immediately begin to rotate unsteadily. We then examine in detail the qualitative and quantitative evolution of the structures as they evolve, and for each case show the sequence of topological bifurcations that occur both in a fixed reference frame, and in an appropriately chosen rotating reference frame. Comparisons between the cases help to reveal different features of the viscous evolution for short and intermediate time scales of vortex structures. We examine the dynamical evolution of passive particles in the viscously evolving flows and interpret it in relation to the evolving streamline patterns. Although the
low-order multi-Gaussian model does not exactly coincide with the Navier-Stokes solution, the two results show remarkable resemblances in many aspects.

In Part II, we examine the effects of body geometry and elasticity on the passive stability of motion in a perfect fluid. Our main motivation is to understand the role of body elasticity on the stability of fish swimming. The fish is modeled as an articulated body made of $N$ links (assumed to be identical ellipses in 2D or identical ellipsoids in 3D) interconnected by hinge joints. It can undergo shape changes by varying the relative angles between the links. Body elasticity is accounted for via the torsional springs at the joints. The unsteadiness of the flow is modeled using the added mass effect. Equations of motion for the body-fluid system are derived using Newtonian and Lagrangian approaches for both hydrodynamically decoupled and coupled models in 2D and 3D. We specifically examine the stability associated with a relative equilibrium of the equations, traditionally referred to as the “coast motion” (proved to be unstable for a rigid elongated body model), and found that body elasticity does stabilize the system. Stable regions are identified based on linear stability analysis in the parameter space spanned by aspect ratio (body geometry) and spring constants (muscle stiffness), and the findings based on the linear analysis are verified by direct numerical simulations of the nonlinear system. Our result shows that even in the absence of fin movement and vortex interaction, shape change and muscle elasticity allow stable conditions of the coast motion.
Part I

Viscous Evolution of Point Vortex Equilibria
Chapter 1

Introduction

The evolution of vortices in the incompressible fluid has long been a fascinating topic in fluid dynamics, particularly in recent decades, thanks to the remarkable developments in experimental techniques and powerful computing resources, which enabled researchers to implement various numerical methods to solve complicated problems. A great number of researches have focused on vortex motion in two-dimensional flow field. It might not be obvious why the understanding of two-dimensional flow is important since we live in a three-dimensional world. In fact, in many applications, such as ocean and atmosphere, one dimension of the flow is much smaller than the other two, and often smaller than the size of the structure of interest in the flow. In such circumstances, a 2D approximation is accurate enough to capture the essence of the physical system.

Studies of vortex motion in 2D date back to Helmholtz [38] in the 19th century, followed by Lord Kelvin [49], Sir Lamb [54], Prandtl [88], Milne-Thomson [77], Batchelor [12], and others. More recent works are summarized in the book by
Saffman [95], which explains general theory of inviscid flow and included brief discussions about viscous flows. The interested readers are also referred to the book by Marchioro & Pulvirenti [65], which discusses mathematical background in inviscid flow; Arnold & Khesin [9], which describes geometrical methods in inviscid flow; and Newton [84], which focuses on theoretical results of multiple point vortices interaction on various 2D manifolds in inviscid flow.

In inviscid 2D flow, point vortex model is commonly used. Point vortex model assumes that the vorticity field is a superposition of \(N\) Dirac delta functions (or point vortices) and everywhere else 0. The flow field is determined from dynamics of the point vortices, which is governed by Euler equation. Vortex system is integrable for three or fewer vortices of any strength, see Novikov [86], Aref [4] and Adam & Ratiu [1]. Dynamics of a single point vortex and two vortex systems are well known. Three vortex systems are also integrable, but their dynamics can be much more complicated. A special family of three vortex systems is the three vortex equilibria (including fixed and relative equilibria), i.e. the vortex structure moves without changing its shape or size, see Aref et al. [8]. Stability of such equilibria is studied in [99] and [6]. If \(N > 3\), the systems are generally proved to be non-integrable by Ziglin [113], Koiller & Carvalho [53] and Castella et al. [18]. For a comprehensive discussion about the \(N\)-vortex problem, the reader is referred to Newton [84].

In the context of viscous 2D flow, due to the nonlinear nature of Navier-Stokes equations, very few problems can be solved analytically, among which most noticeably is the Lamb-Oseen vortex. The Lamb-Oseen vortex has a nonzero circulation
and the center of vortex remains fixed in the absence of ambient flow or outside forcing. The vorticity is a Gaussian distribution spreading in time. Once the initial or boundary conditions become more complicated, in most situations, one can only resort to numerical methods. Several categories of numerical methods have been developed, for instance, finite difference methods, finite element methods [26] and vortex methods, just to name a few. Vortex methods are grid-less methods, based on the idea of discretizing the vorticity field and using Lagrangian instead of Eulerian description of the flow. Historically, development of vortex methods dates back to Rosenhead [91, 92] in the early 1930s, in which point vortex model was used to study unsteady flow, and calculated by hand. Modern vortex methods originated in the works of Chorin [20] and Milinazzo & Saffman [76], in which the random-walk technique was used in numerical study of diffusion. Aided by the power of computers, studies of vortex method schemes and theoretical proofs of convergence of such schemes quickly became very active, and the reader is referred to, for instance, the survey paper by Leonard [57], the convergence proofs by Goodman [32] and Hou [39] and the comprehensive book by Cottet & Koumoutsakos [22]. A family of methods called core growing methods treats the cores of vorticity as Gaussian distributions which are spreading in time, see, for instance, Kida & Nakajima [51]. However, as Greengard [33] pointed out, this algorithm converges to a system of equations that are different from Navier-Stokes equations. Several alternative core growing approaches managed to overcome such problem, see, for example, Lu & Ross [62], Cottet & Koumoutsakos [22] and Rossi [93]. Though long time behavior predicted
by the original core growing method may converge to the dynamics governed by Navier-Stokes equations, it can very well approximate the widely separated regions of vorticity for a long period of time (see, e.g. the experimental and numerical work by Meunier et al. [74]), or approximate the general vorticity field for a relatively short period of time.

When point vortex equilibria are used as initial conditions for the corresponding Navier-Stokes equations in an unbounded flow field, an interesting and complex dynamical process unfolds at short and intermediate time scales, which obviously depends crucially on the details of the initial configuration. For long enough times, Gallay & Wayne proved recently that the Lamb-Oseen solution is an asymptotically stable attracting solution for all integrable initial vorticity fields [31]. Although very powerful, this asymptotic result does not elucidate the intermediate dynamics that takes place in finite time and allows a given initial vorticity field to reach the single peaked Gaussian distribution of the Lamb-Oseen solution. Therefore, an analysis of how these equilibria evolve under the full Navier-Stokes equation should merit a systematic treatment. In this work, we begin an investigation of the viscous evolution of point vortex equilibria using a core growing model referred to as the multi-Gaussian model. Four configurations are studied as examples to examine the dynamics in short and intermediate time scales.

Two examples of 2-vortex configurations are examined: vorticity ratio 1 : 1 and 2 : 1, they are both co-rotating cases with nonzero total vorticity. These two cases are examples of the much studied symmetric and asymmetric vortex mergers. The
merging of two co-rotating vortices is an essential phenomenon in transitional and turbulent flows. Earlier studies of vortex mergers were set in inviscid flow. Dritschel & Waugh [23] and Trieling et al. [101] considered both symmetric and asymmetric vortex pairs, and identified 5 flow regimes of vortex interaction, namely, elastic interaction, partial straining-out, complete straining-out, partial merger and complete merger. Trieling et al. also verified their analytical results with experiments, using dye (passive particles) visualization, see Figure 1.1. However, in viscous flow, vorticity field always approaches a single Gaussian distribution, hence, the interaction of vortex pairs should always result in a “complete merger”. Recently, more studies have focused on vortex pair interaction in viscous flow. For the symmetric case, Melander et al. [72] examined the flow structure in a co-rotating frame, and identified flow regions known as inner cores, exchange band and outer recirculation

Figure 1.1: Dye visualization of (a) symmetric and (b) asymmetric co-rotating vortex pair merging process from Trieling et al. [101].

(a) Symmetric vortex merger

(b) Asymmetric vortex merger
regions. Cerretelli & Williamson [19] and Meunier et al. [74] described the merging process in three stages, namely, the first diffusive, convective and second diffusive stage, and associated the “onset” of merging with the beginning of convective stage. They also associated the filaments as the primary driving force of the merging process. However, Velasco Fuentes [27] found that filaments are not the reason for the merging, instead just the result of it. Fewer studies have focused on the asymmetric case, Brandt & Nomura [16] classified regimes of interaction based on time scales, and associated merging with the strain rate of velocity field. Determination of a merging criterion for the vortex pairs has been the focus of many studies both in inviscid and viscous setups, mostly for the symmetric case [96, 87, 73, 15], and fewer for the asymmetric case [23, 101, 16]. For the symmetric case, the “onset” of merging is usually determined by a critical ratio between the vortex core size $a$ and the distance between vortex centers $b$. The critical ratio $(a/b)_{cr}$ has been found to be between 0.23 and 0.31 in the mentioned studies. For the asymmetric case, whether a universal critical ratio can be found still remains an open question.

Also, two cases of collinear 3-vortex configurations are examined, and the strength ratios are $-1 : 2 : -1$ and $2 : -1 : 2$. The first case is traditionally known as a tripole, and its total circulation is zero. Experimental works on tripole have been conducted by Swenson [98], van Heijst & Kloosterziel [104], Carton et al. [17] and van Heijst et al. [105]. Numerical evidence of the existence of tripoles was found by Legras et al. [56], which, interestingly, was before the confirmation of the existence of tripoles in experiments. Recently, there have been more numerical
studies of tripole. For instance, Rossi et al. [94] and Barba & Leonard [11] both discussed about the generation of a tripole by perturbing a monopole (Lamb-Oseen vortex). Besides experimental and numerical works, there have been a few analytical works aimed to model the evolution of tripoles, for example, van Heijst & Kloosterziel [104] presented a model based on a point vortex setup, but the strengths of point vortices were altered as functions of time in order to fit the experimental result; Kizner & Khvoles [52] found inviscid explicit solution for the region away from the vortices. However, none of these models addressed the problem in a viscous fluid, which can dramatically affects the results. The second case (2 : −1 : 2) is much less studied in the literature. It turns out the collinear state shows much richer dynamics than a tripole using the multi-Gaussian model.

The remaining of Part I is organized in the following way: Chapter 2 provides a brief overview the background knowledge of incompressible fluids including the inviscid point vortex model and the Lamb-Oseen vortex solution in viscous flow.
The multi-Gaussian model is then presented and proposed as a low-order model for viscously-evolving point vortex equilibria in a general context. In Chapter 3, we study the four examples mentioned above. For each case, we analyze the evolution of the flow field observed in a co-rotating frame that rotates with the same rate of the structure. We find instantaneous stagnation points and analyze the associated eigenvalue problems. We also find bifurcation sequence of the streamlines. In Chapter 4, we summarize the current work.
Chapter 2

Multi-Gaussian Model

2.1 Background

In general, for three-dimensional incompressible flow, velocity $u$ of the flow is governed by *Navier-Stokes* equation

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = -\frac{1}{\rho} \nabla p + \nu \Delta u,$$

(2.1)

and continuity equation

$$\nabla \cdot u = 0,$$

(2.2)

where $\rho$ is the fluid density (a constant due to incompressibility), $p$ is pressure, and $\nu$ is kinematic viscosity (assumed to be constant). *Vorticity* $\omega$ is defined as the curl of velocity

$$\omega = \nabla \times u.$$  

(2.3)
Upon taking the curl of the velocity equation (2.1), one obtains Navier-Stokes equation in terms of vorticity

$$\frac{\partial \omega}{\partial t} + \mathbf{u} \cdot \nabla \omega = \omega \cdot \nabla \mathbf{u} + \nu \Delta \omega,$$

(2.4)

where the fact $\nabla \times \nabla p = 0$ is already imposed. Also, continuity equation for vorticity is given by

$$\nabla \cdot \omega = 0.$$

(2.5)

Velocity $\mathbf{u}(\mathbf{x}, t)$ and vorticity $\omega(\mathbf{x}, t)$ are both functions of position vector $\mathbf{x}$ and time $t$. They can be expressed in an inertial frame $\{e_i\}_{i=1,2,3}$, that is to say, one has $\mathbf{x} = x \mathbf{e}_1 + y \mathbf{e}_2 + z \mathbf{e}_3$, $\mathbf{u} = u_x \mathbf{e}_1 + u_y \mathbf{e}_2 + u_z \mathbf{e}_3$ and $\omega = \omega_x \mathbf{e}_1 + \omega_y \mathbf{e}_2 + \omega_z \mathbf{e}_3$, or, equivalently, $\mathbf{x} = (x, y, z)$, $\mathbf{u} = (u_x, u_y, u_z)$ and $\omega = (\omega_x, \omega_y, \omega_z)$. Note that $\mathbf{u}$ is still present in the vorticity equation (2.4). One can compute velocity in terms of vorticity by the Biot-Savart law in $\mathbb{R}^3$

$$\mathbf{u}(\mathbf{z}) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{(\mathbf{x} - \bar{\mathbf{x}}) \times \omega(\bar{\mathbf{x}})}{\|\mathbf{x} - \bar{\mathbf{x}}\|^3} \, d\bar{\mathbf{x}},$$

where $\bar{\mathbf{x}}$ is an integration variable. A fundamental scaler quantity associated with the vorticity is circulation $\Gamma$, which is defined as

$$\Gamma = \oint_C \mathbf{u} \cdot d\mathbf{s} = \int_A \mathbf{\omega} \cdot \mathbf{n} \, d\sigma,$$
where $A$ is any open surface bounded by the closed curve $C$, $n$ is a unit normal vector of the surface. Physically, circulation is the flux of vorticity through an surface.

For two-dimensional incompressible flow, $\omega$ is always perpendicular to the plane of dynamics, hence the stretching term $\omega \cdot \nabla u$ is always zero. The continuity equation (2.5) is automatically satisfied. The Navier-Stokes equation (2.4) is reduced to a scaler equation

$$\frac{\partial \omega}{\partial t} = -u \cdot \nabla \omega + \nu \Delta \omega,$$  

(2.6)

where $\omega$ is the only nonzero component of $\omega$ in the 2D flow. Velocity can be computed by the Biot-Savart law in $\mathbb{R}^2$

$$u(z) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x - \tilde{x})^\perp}{\|x - \tilde{x}\|^2} \omega(\tilde{x}) d\tilde{x},$$  

(2.7)

where $x = (x, y)$ is the position vector in a 2D inertial frame, and $x^\perp = (-y, x)$.

We will deal exclusively with 2D incompressible flow in this work. Traditionally, problems in 2D can be expressed compactly using complex notation with position variable $z$, where $z = x + iy$ and $i = \sqrt{-1}$.

If one is interested in inviscid flow, i.e. $\nu = 0$, the diffusion term $\nu \Delta \omega$ is zero, then (2.6) reduces to Euler equation

$$\frac{\partial \omega}{\partial t} = -u \cdot \nabla \omega.$$  

(2.8)
And if pure diffusion problem is concerned, i.e. the convection term $u \cdot \nabla \omega$ is zero, then (2.6) becomes the heat equation

$$\frac{\partial \omega}{\partial t} = \nu \Delta \omega.$$  \hfill (2.9)

\section{2.2 Point vortex model}

In this section, we limit ourselves to considering only \textit{inviscid} unbounded fluid. Governing equation for vorticity is given by the Euler equation (2.8), and velocity can be computed from the Biot-Savart law in 2D (2.7). If vorticity in the fluid is assumed to be always concentrated at $N$ \textit{point vortices}, and everywhere else 0, the vorticity distribution can be written as

$$\omega(z) = \sum_{\alpha=1}^{N} \Gamma_{\alpha} \delta(z - z_{\alpha}),$$  \hfill (2.10)

where $\delta$ is the Dirac delta function, $z_{\alpha}$ is the position of the $\alpha^{\text{th}}$ point vortex and $\Gamma_{\alpha}$ is the circulation of the $\alpha^{\text{th}}$ vortex. Since there is no diffusion, vorticity of the point vortices will remain constant for all time.

As a special case, the vorticity field of a \textit{single} isolated point vortex with circulation $\Gamma$ located at the origin is given by

$$\omega(z) = \Gamma \delta(z - z_{1}), \quad z_{1} = 0 + i0.$$  \hfill (2.11)
To obtain the velocity field around a point vortex, one can substitute (2.11) into the Biot-Savart law (2.7), and write the equation in Cartesian coordinates

\[
\begin{align*}
    u_x(x, y) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{-(y - \tilde{y})}{(x - \tilde{x})^2 + (y - \tilde{y})^2} \Gamma \delta(\tilde{x}) \delta(\tilde{y}) d\tilde{x} d\tilde{y} = \frac{\Gamma}{2\pi} \frac{-y}{x^2 + y^2} , \\
    u_y(x, y) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{(x - \tilde{x})}{(x - \tilde{x})^2 + (y - \tilde{y})^2} \Gamma \delta(\tilde{x}) \delta(\tilde{y}) d\tilde{x} d\tilde{y} = \frac{\Gamma}{2\pi} \frac{x}{x^2 + y^2} ,
\end{align*}
\]

where \( \tilde{x} \) and \( \tilde{y} \) are integration variables. Equivalently, velocity can be compactly expressed in complex variable,

\[
\mathbf{u}^* \equiv u_x - iu_y = \mathbf{z}^* = \frac{\Gamma}{2\pi i} \frac{1}{z} ,
\]

where \( \mathbf{z} = x + iy \) is the position variable, and the superscript \((\cdot)^*\) indicates complex conjugate. From (2.13), one obtains the complex conjugate of velocity, called complex velocity (sometimes also simply referred to as velocity). Notice that the velocity field is undefined at the center of the point vortex. Since a point vortex does not induce velocity on itself, the position of an isolated point vortex is fixed, i.e. \( \mathbf{z}_1 \equiv 0 + i0 \) for all time.

However, for \( N \) point vortices \( (N > 1) \), this will not be true. Instead, positions of the vortices are dictated by the local velocity field induced by the presence of all the other vortices. Fluid velocity at an arbitrary point that does not coincide with
a point vortex is obtained from the Biot-Savart law (2.7) which, following a similar
derivation as in (2.12), takes the form

\[ \dot{z}^* = \sum_{\alpha=1}^{N} \frac{\Gamma_\alpha}{2\pi i} \frac{1}{z - z_\alpha}. \]  

(2.14)

whereas the velocity of a point vortex \( z_\beta \) is given by subtracting the effect of that
point vortex from (2.14), and replacing \( z \) with \( z_\beta \), namely,

\[ \dot{z}^*_\beta = \sum_{\alpha \neq \beta}^{N} \frac{1}{2\pi i} \frac{\Gamma_\alpha}{z_\beta - z_\alpha}. \]  

(2.15)

The \( 2N \) first-order ordinary differential equations (2.15) dictating the inviscid evo-
lution of \( N \) point vortices are known to exhibit regular (including fixed and relative
equilibria) as well as chaotic dynamics depending on the number of vortices, their
strengths and initial positions. The literature on this general topic is large, and we
refer simply to the influential review article of Aref [5], along with the monographs
of Saffman [95] (especially Chapter 7) and Newton [84] for an immediate entry into
the literature. We also mention the 2008 IUTAM Symposium “150 Years of Vortex
Dynamics” in which the lively state-of-the-art developments were reported [7].

### 2.3 Examples of point vortex dynamics

We now examine some examples of the dynamics of multiple point vortices.
Dipole A vortex dipole consists of two point vortices of equal and opposite strengths, as depicted in Figure 2.1. Denote the positions of the left and right vortices in complex variables as $z_L$ and $z_R$, respectively, and they are separated by a distance $b_0$. The circulation of $z_L$ ($z_R$) is $\Gamma$ ($-\Gamma$). Note the net circulation in the flow field is zero. Initially,

$$z_L(0) = \frac{b_0}{2} + i0, \quad z_R(0) = -\frac{b_0}{2} + i0.$$ 

Substitute positions and strengths into (2.15), velocities of the vortices are given by

$$\dot{z}_L^* = \dot{z}_R^* = 0 - i \frac{\Gamma}{2\pi b_0}.$$ 

Therefore, positions of the vortices as functions of time are given by

$$z_L = \frac{b_0}{2} + i \frac{\Gamma}{2\pi b_0} t, \quad z_R = -\frac{b_0}{2} + i \frac{\Gamma}{2\pi b_0} t,$$

that is, the vortices move along parallel straight lines with constant velocities.
Co-rotating pair A co-rotating vortex pair consists of two point vortices of the same sign, but their strengths may be different, see Figure 2.2. Net circulation in the flow field is not zero. As before, the positions of the left and right vortices are denoted as \( z_L \) and \( z_R \), respectively, and they are separated by \( b_0 \). Without loss of generality, circulations of both vortices are assumed to be counterclockwise (positive), and the strengths of the vortices are denoted as \( \Gamma_L \) and \( \Gamma_R \), respectively. One can describe the dynamics in an inertial frame with \( x \) axis parallel to the line connecting the two vortices, and the origin at \( O \) between the vortices, such that if the distance between \( z_L \) (\( z_R \)) and \( O \) is denoted as \( r_L \) (\( r_R \)), one has the following relation:

\[
\frac{\Gamma_L}{\Gamma_R} = \frac{r_R}{r_L} \Rightarrow r_L = \frac{\Gamma_R}{\Gamma_L + \Gamma_R} b_0, \quad r_R = \frac{\Gamma_L}{\Gamma_L + \Gamma_R} b_0, \quad (2.16)
\]

since \( r_L + r_R = b_0 \). Initially the positions are \( z_L(0) = -r_L + i0, z_R(0) = r_R + i0 \). Since the induced velocities from these two point vortices onto each other are only perpendicular to the line connecting the vortices, distance between the two vortices remains constant \( b_0 \) for all time, and the two vortices are co-rotating at a constant rotation rate \( \dot{\theta} \) with the center of rotation located at \( O \) and distances from the
origin being constants for all time. Hence, dynamics of this vortex pair is entirely determined by this constant rotation rate. To obtain $\dot{\theta}$, we will focus on the motion of one of the vortex, say, $z_R$. It is more convenient to describe the system in polar coordinates, where positions of the vortices can be represented in exponential form

$$z_R = R e^{i\theta}, \quad z_L = -r_L e^{i\theta}.$$  \hfill (2.17)

Therefore the complex velocity of $z_R$ is given by

$$\dot{z}_R^* = \left(\dot{R} e^{i\theta} + r_R i \dot{\theta} e^{i\theta}\right)^* = -ir_R \dot{\theta} e^{-i\theta},$$  \hfill (2.18)

since $r_R$ is constant. On the other hand, velocity of the right vortex is induced by the left vortex. Substitute positions and strengths into (2.15),

$$\dot{z}_R^* = \frac{1}{2\pi i} \frac{\Gamma_L}{z_R - z_L} = \frac{1}{2\pi i} \frac{\Gamma_L}{b_0 e^{i\theta}}.$$  \hfill (2.19)

Equating (2.18) and (2.19), the rotation rate is given by

$$\dot{\theta} = \frac{\Gamma_L + \Gamma_R}{2\pi b_0^2}.$$  \hfill (2.20)

And the rotation angle is given by $\theta = \dot{\theta} t$. Note that the rotation rate depends only on the total circulation $\Gamma_L + \Gamma_R$ of the flow and the distance $b_0$ between the two vortices, but not on the strength ratio of the two vortices. As a special case, if the strengths of both vortices are the same, it is called a symmetric co-rotating
pair, see Figure 2.2(a); otherwise, it is an asymmetric co-rotating pair, an example is depicted in Figure 2.2(b). The two examples have the same total circulation and separation, hence their rotation rates are the same, \( \dot{\theta} = \Gamma / \pi b_0^2 \).

![Figure 2.3: Schematics of a vortex tripole.](image)

**Tripole** Traditionally, the term *tripole* refers to a special case of three vortex collinear state such that the three vortices are equally spaced on a straight line with distances between the adjacent vortex centers being \( b_0 \), and the strength ratio of these three vortices is \(-1 : 2 : -1\) from left to right, see Figure 2.3. Net circulation of the flow field is zero in this case. Due to symmetry, the three vortices will always remain on a straight line, and the distances between the two adjacent vortices will also remain constants. In another word, the tripole configuration is a relative equilibrium. But it is not a fixed equilibrium because the tripole will rotate about the center vortex at a constant rotation rate. Without loss of generality, denote the strengths of the left, center and right vortices as \(-\Gamma, 2\Gamma\) and \(-\Gamma\), respectively. Positions of the vortices are denoted as \( z_L, z_C \) and \( z_R \), respectively, in an inertial frame originated at the center vortex, and \( x \) axis parallel to the line initially connecting the vortices. One has \( z_C \equiv 0 + i0 \) for all time, and \( z_L = -z_R = -b_0 e^{i\theta} \).
Similar to the co-rotating pair, we will focus on velocity of the right vortex \( z_R \) in order to obtain the rotation rate \( \dot{\theta} \). The exponential form of the right vortex is \( z_R = b_0 e^{i\theta} \), hence the complex velocity is given by

\[
\dot{z}_R^* = -ib_0 \dot{\theta} e^{-i\theta},
\]

(2.21)

also from the induced velocity point of view

\[
\dot{z}_R^* = \frac{1}{2\pi i} \left( \frac{-\Gamma}{2b_0 e^{i\theta}} + \frac{2\Gamma}{b_0 e^{i\theta}} \right).
\]

(2.22)

Equating these two, the rotation rate is given by

\[
\dot{\theta} = \frac{3\Gamma}{4\pi b_0^2}.
\]

(2.23)

Figure 2.4: Schematics of a three vortex collinear fixed equilibrium.

**Three vortex collinear fixed equilibrium**  This is another special cases of three vortex collinear state. The spatial configuration of this case is the same tripole, but the strength ratio is \( 2 : -1 : 2 \), see Figure 2.4. In another word, net circulation of the flow field is *not* zero. Obviously, due to symmetry, this case is at least
a relative equilibrium. To prove it is indeed a fixed equilibrium, we examine the induced velocities for each vortex. For the center one $z_C$, induced velocities from the left and right vortices cancel out, and $\dot{z}_C = 0$. The left and right vortices are symmetric, hence we only need to examine one of them, say, $z_R$:

$$\dot{z}_R = \frac{1}{2\pi i} \left( \frac{2\Gamma}{2b_0e^{i\theta}} + \frac{-\Gamma}{b_0e^{i\theta}} \right) = 0.$$ 

Therefore, velocities of the side vortices is also zero, and the structure is a fixed equilibrium. In another word, the rotation rate is $\dot{\theta} = 0$ for all time.

### 2.4 The Lamb-Oseen vortex

We consider the evolution of an initial single point vortex in a viscous unbounded fluid. Due to axisymmetry, there is no preferable direction of movement of the vortex. In another word, the vortex does not induce velocity onto itself, the position of the vortex center remains fixed for all time. Therefore, this is a pure diffusion problem, and vorticity is governed by the heat equation (2.9). Without loss of generality, assume the vortex center is located at the origin $O$ and circulation is $\Gamma$. The vorticity field at time $t = 0$ is given by

$$\omega(z, 0) = \Gamma \delta(z).$$
The solution is the well-known *Lamb-Oseen vortex* [54, 12, 95]. As noted by Saffman [95], there are various ways to solve this problem. For instance, taking a Laplace transformation in time, or a Fourier expansion in space, or dimensional analysis, they all work equally well. Mathematically, this is a solution of heat equation. For details of the solution, see, e.g. Haberman [34]. The vorticity field is a function of both space and time,

\[ \omega(z, t) = \frac{\Gamma}{4\pi \nu t} \exp \left( -\frac{\|z\|^2}{4\nu t} \right). \tag{2.24} \]

Due to axisymmetry, if the vorticity is expressed in polar coordinates \((r, \theta)\), it does not depend on \(\theta\),

\[ \omega(r, t) = \frac{\Gamma}{4\pi \nu t} \exp \left( -\frac{r^2}{4\nu t} \right). \]

The velocity field associated with the Lamb-Oseen vortex is also axisymmetric, i.e. radial velocity \(v_r = 0\), and tangential velocity \(v_\theta\) can be calculated using the Biot-Savart law,

\[ v_\theta(r, t) = \frac{1}{2\pi} \int_0^{2\pi} \int_0^r \frac{\omega(\tilde{r}, t)}{r} \tilde{r} \, d\tilde{r} \, d\tilde{\theta} = \frac{\Gamma}{2\pi r} \left[ 1 - \exp \left( -\frac{r^2}{4\nu t} \right) \right], \]

where \(\tilde{r}\) and \(\tilde{\theta}\) are integration variables in the \(r\) and \(\theta\) directions, respectively. Equivalently, velocity can be written compactly in complex notation,

\[ \dot{z}^* = \frac{\Gamma}{2\pi i} \frac{1}{z} \left[ 1 - \exp \left( -\frac{\|z\|^2}{4\nu t} \right) \right]. \tag{2.25} \]
Figure 2.5: (a) Vorticity $\omega$ and (b) tangential velocity $v_\theta$ as functions of radius $r$ and time $t$ (not functions of angle $\theta$ due to axisymmetry).

Vorticity and tangential velocity are depicted as functions of $r$ in Figure 2.5 for various instants. One can compare the vorticity result to a heat problem, whose temperature is governed by the heat equation with initial Dirac delta heat source

$$\frac{\partial T}{\partial t} = \nu \Delta T, \quad T(z, 0) = \Gamma \delta(z).$$

The solution to above equation is a Gaussian distribution

$$T(z, t) = \frac{\Gamma}{4\pi\nu t} \exp \left( -\frac{\|z\|^2}{4\nu t} \right).$$

The vorticity field for Lamb-Oseen vortex has exactly the same form, therefore, a Lamb-Oseen vortex is also referred to as a Gaussian vortex. Evolution of vorticity according to (2.24) is the following: vorticity is concentrated at the origin at $t = 0$, as soon as $t > 0$, vorticity diffuses axisymmetrically as a Gaussian distribution due to viscosity. The spreading of concentration can be quantified by the so-called
vortex core, which is defined as $a = \sqrt{4\nu t}$. Comparing the Lamb-Oseen vortex to the point vortex, one might be tempted to view the Lamb-Oseen vortex as a point vortex with time dependent circulation. However, this is incorrect because such circulation would be not only a function of time, but also a function of space. Contrary to the inviscid flow, in the viscous case energy is not conserved and the system is not Hamiltonian. The system cannot be characterized by a complex stream function. For a discussion about the Lamb-Oseen vortex in detail, one is referred to, for example, Batchelor [12].

2.5 Multi-Gaussian model

In Section 2.2, we discussed the dynamics of $N$ inviscid point vortices, while in Section 2.4, we studied the evolution of an isolated single point vortex in viscous flow. The following question naturally arises: given an initial condition of $N$ point vortices, in a viscous fluid, how does the flow field evolve? Mathematically, we want to solve equation (2.6) and (2.7) subject to the following initial condition:

$$\omega(z, 0) = \sum_{\alpha=1}^{N} \Gamma_\alpha \delta(z - z_\alpha). \quad (2.26)$$

As stated before, this is not an easy problem, and typically does not have analytical solution. One can solve this problem using numerical techniques, such as vortex methods, see for example Cottet & Koumoutsakos [22], Leonard [57] and the subsequent analysis by Majda & Bertozzi [63]. One example of vortex methods is
a “time splitting” scheme, which can be traced at least back to Chorin’s influential paper [20], also used by Milinazzo and Saffman [76]. In these papers, the splitting scheme can be summarized as follows: in the first sub-step, vortex centers are convected by solving Euler equation (2.8); in the second sub-step, effect of diffusion due to viscosity is accounted for by an execution of a random-walk, whose walk step size is \( \sqrt{4\nu\Delta t} \), with \( \Delta t \) being the time increment. This algorithm may also involve several sub-steps in order to increase accuracy. One difficulty of the time splitting scheme is the blow-up of inviscid velocities when two vortices come too close to each other. Hence, a concept of vortex blob is introduced to overcome such difficulty: replace the velocity given in (2.15) by a piecewise continuous function \( u^*_j \) such that, when vortices are far from each other, \( u^*_j \) behaves like a point vortex, and when the distance \( r \) between two of the vortices becomes small enough, say, \( r < \epsilon \), \( u^*_j \) behaves not like \( 1/r \), but like \( r/\epsilon \) [20] or \( r^2/\epsilon^2 \) [76], etc. Such \( \epsilon \) defines a “blob” around the center of vortex. Notice the choices of both the size of the blob and the velocity profile inside the blob are rather ad hoc. Another type of vortex method is to let the blob grow in time, most commonly \( \epsilon \sim \sqrt{\nu t} \) (a spreading Gaussian core), and the vortex centers move according to their local velocity field induced by other Gaussian cores instead of the point vortex velocity field. Therefore, the randomness is eliminated. Such approach is called a core spreading or a core growing method. However, Greengard [33] pointed out that the core growing method does not converge to a Navier-Stokes solution. Several alternative core growing approaches have
already been developed to overcome this disadvantage. Focus studies using numerical simulations can be found in the works of Barba & Leonard [10, 11], and used in specific models in Meiburg & Newton et al. [70, 85, 71]. We mention, of course, also the works of Gallay & Wayne [29, 30, 31], and the Ph.D. thesis of Uminsky [103] and the follow-up work [81] which developed an eigenfunction expansion method based on the form of heat-kernel. Additionally, we mention the body of work generated by Dritschel and co-workers, of which [66, 90] would be two relevant examples, whose aim is to elucidate via Lagrangian type of numerical simulations, the host of complex processes associated with mixing and dynamics in viscously evolving two-dimensional flows. Nevertheless, the original core growing method approximates the flow field very well when the vorticity regions are far apart or the time period is short. Typically, when vortex methods are used, the number of vortices fed into the flow field should be as large as possible in order to achieve higher accuracy.

Instead of solving the problem numerically, we implement a simple, analytically tractable model to describe the dynamical evolution of $N$ initial point vortices for nonzero viscosity, which is inspired by the idea of the original core growing vortex method. The model assumes the vorticity of each vortex to spread axisymmetrically as an isolated Lamb-Oseen vortex, thus accounting for the diffusion term $\nu \Delta \omega$ in (2.6). In another word, the vorticity field in this model is given by the linear superposition of $N$ Gaussian vortices,

$$\omega(z, t) = \sum_{\alpha=1}^{N} \frac{\Gamma_{\alpha}}{4\pi \nu t} \exp \left( -\frac{\|z - z_{\alpha}\|^2}{4\nu t} \right). \quad (2.27)$$
The associated velocity field can be computed by substituting (2.27) into (2.7),

\[ \dot{z}^*(\mathbf{z}, t) = \sum_{\alpha=1}^{N} \frac{\Gamma_{\alpha}}{2\pi i(\mathbf{z} - \mathbf{z}_{\alpha})} \left[ 1 - \exp \left( \frac{-\|\mathbf{z} - \mathbf{z}_{\alpha}\|^2}{4\nu t} \right) \right]. \quad (2.28) \]

And the centers move according to the local velocity induced by the presence of other diffusing vortices, thus accounting for the convection term \( \mathbf{u} \cdot \nabla \omega \) in (2.6),

\[ \dot{z}_\beta = \sum_{\alpha \neq \beta}^{N} \frac{\Gamma_{\alpha}}{2\pi i(\mathbf{z}_\beta - \mathbf{z}_{\alpha})} \left[ 1 - \exp \left( \frac{-\|\mathbf{z}_\beta - \mathbf{z}_{\alpha}\|^2}{4\nu t} \right) \right]. \quad (2.29) \]

It is worth mentioning here the recent work of Gallay [28] in which the author analyzed the inviscid limit \( \nu \to 0 \) of the 2D Navier-Stokes evolution of Dirac delta initial conditions, and proves, under certain assumptions, that the solution of the Navier-Stokes equation converges, as \( \nu \to 0 \), to a superposition of Lamb-Oseen Vortices. We refer to the model given by (2.27)~(2.29) as the \textit{multi-Gaussian model}, and all three equations are crucial to the understanding of the evolution. We specifically emphasize that including (2.28) in this model paves the way for the evolution of passive particles in the flow field which is transported under the dynamics generated by (2.27) and (2.29). This will be discussed more thoroughly in the following sections. A schematic illustration of the model is shown in Figure 2.6.

As stated before, (2.27) is not an exact solution of the Navier-Stokes equation in (2.6), because the cores of vortices do not remain axisymmetrical when (2.6) is solved exactly. Instead, they deform according to the local velocity field. The
multi-Gaussian model is accurate when the vortices are far away from each other, i.e. the cores $a_\beta \ll \|z_\beta - z_\alpha\|, \forall \alpha \neq \beta$. Here, the multi-Gaussian model is not intended to be a numerical approach; it does not feed the flow field with very large number of vortices. Instead it only assumes Gaussian spreading cores located initially at the vorticity peaks. By employing the multi-Gaussian model, the problem becomes solving a few nonlinear ODEs instead of the nonlinear Navier-Stokes equation, therefore enabling us to utilize the powerful tools developed for dynamical systems.

We now compare the multi-Gaussian model to the heat equation subject to $N$ Dirac delta initial sources,

$$\frac{\partial T}{\partial t} = \nu \Delta T, \quad T(z, 0) = \sum_{\alpha=1}^{N} \Gamma_\alpha \delta(z - z_\alpha),$$
whose solution takes the form

\[ T(z, t) = \sum_{\alpha=0}^{N} \frac{\Gamma_\alpha}{4\pi\nu t} \exp \left( -\frac{\|z - z_\alpha\|^2}{4\nu t} \right). \]

Although the vorticity (2.27) appears to have the same form as the temperature distribution given above, they are fundamentally different, and the difference is the positions of centers. In the heat problem, positions of heat source centers are fixed, hence the solution is merely a linear superposition of solutions of isolated heat source. However, in the fluid problem, vortex centers will generally not remain stationary, instead they will be advected by the local velocity field according to (2.29).
Chapter 3

Case Studies - Viscous Evolution of Vortex Structures

We now use the multi-Gaussian model to analyze the fluid velocity and vorticity fields at intermediate time scales before the asymptotic state of a single Lamb-Oseen vortex dominates. The goal of this analysis is to understand the intermediate mechanisms that lead the initial state to the final state. Since the details of such process depends on initial conditions, such analysis must be conducted in a case by case fashion. We will present four examples of different initial conditions: Section 3.1 discusses the symmetric co-rotating pair, Section 3.2 discusses the asymmetric co-rotating pair, Section 3.3 discusses the vortex tripole and Section 3.4 studies the collinear three vortex fixed equilibrium. We start with the symmetric co-rotating pair because it is the most straight forward, and most studied case in the literature. All detailed steps are presented in this section. The following sections only show the key steps, and the reader can refer to the symmetric co-rotating pair case for similar procedures. The asymmetric case is the only asymmetric example shown in
this study. One of the purposes of this section is to demonstrate both the common
and different phenomenon in the symmetric and asymmetric setups. Vortex tripole
is a special case among these four examples because it is the only case that has a
zero net circulation in the flow field, therefore the conclusion of Gallay & Wayne
does not apply for tripole [31]. However, we can show the evolution of the flow
field using the model. The collinear three vortex fixed equilibrium is a case that
illustrates the riches dynamics among these four. We will show comparison between
multi-Gaussian model and direct numerical simulation in order to demonstrate the
strength and error of the multi-Gaussian model.

It is a good time to nondimensionalize the problem using the length scale $b_0$
(which is the distance between adjacent vortices), and time scale $T = b_0^2/\Gamma$ ($\Gamma$ is a
typical circulation of vortex). Therefore, the variables are nondimensionalized as

$$
\tilde{r} = \frac{r}{b_0}, \quad \tilde{t} = \frac{t}{T}, \quad \tilde{\Gamma} = \frac{\Gamma}{b_0^2/T} = 1, \quad \tilde{\nu} = \frac{\nu}{b_0^2/T} = \frac{\nu}{\Gamma} = \frac{1}{Re},
$$

$$
\tilde{\mathbf{u}} = \frac{\mathbf{u}}{b_0/T}, \quad \tilde{\omega} = \frac{\omega}{1/T},
$$

where the Reynolds number is usually defined as $Re = \Gamma/\nu$ in viscous vortex dy-
namics. Navier-Stokes equation is thus written as

$$
\frac{\partial \tilde{\omega}}{\partial \tilde{t}} = -\tilde{\mathbf{u}} \cdot \tilde{\nabla} \tilde{\omega} + \frac{1}{Re} \tilde{\Delta} \tilde{\omega}.
$$
The multi-Gaussian model is nondimensionalized as well

\[ \tilde{\omega} = \sum_{\alpha=1}^{N} \frac{\tilde{\Gamma}_{\alpha}}{4\pi t/\text{Re}} \exp \left( -\frac{\|\tilde{z} - \tilde{z}_{\alpha}\|^2}{4t/\text{Re}} \right). \]

The velocity field becomes

\[ \tilde{\dot{z}}^* = \sum_{\alpha=1}^{N} \frac{\tilde{\Gamma}_{\alpha}}{2\pi i(\tilde{z} - \tilde{z}_{\alpha})} \left[ 1 - \exp \left( -\frac{\|\tilde{z} - \tilde{z}_{\alpha}\|^2}{4t/\text{Re}} \right) \right]. \]

One can see that the nondimensional expressions have the same form as the dimensional ones, furthermore, viscosity \( \nu \) can be replaced by \( 1/\text{Re} \). One can notice that \( \tilde{t} \) is always multiplied by \( 1/\text{Re} \) in the equations, hence it is more convenient to define a new nondimensional variable, or a scaled time,

\[ \tilde{\tau} = \frac{\tilde{t}}{\text{Re}}. \]

For simplicity, we will drop the \( \tilde{()})\) notation with the understanding that all variables are nondimensionalized hereafter. The same nondimensionalization will be implemented for all cases.

### 3.1 Symmetric co-rotating vortex pair

We remind the reader of the inviscid result of symmetric co-rotating pair first: a symmetric co-rotating vortex pair consists of two point vortices of the same strength \( \Gamma \), therefore the net circulation is not zero, at a distance \( b_0 \) from each other.
The system is a relative equilibrium. In dimensionless form, $z_L(0) = -\frac{1}{2} + i0$ and $z_R(0) = \frac{1}{2} + i0$. The pair rotates around the center point at a constant angular velocity

$$\dot{\theta} = \frac{\Gamma}{\pi b_0^2}, \quad (3.1)$$

in dimensionless form $\hat{\theta} = 1/\pi$.

**Dynamics of vortex centers** Implementing the multi-Gaussian model, the dimensionless vorticity field is given by

$$\omega = \frac{1}{4\pi\tau} \left[ \exp \left( -\frac{\|z - z_L\|^2}{4\tau} \right) + \exp \left( -\frac{\|z - z_R\|^2}{4\tau} \right) \right]. \quad (3.2)$$

Dynamics of vortex centers is determined as follows: similar to the inviscid case, the induced velocities of the vortex centers are always perpendicular to their connecting line, therefore the distance between the vortex centers is a constant, and the system rotates around the center point due to symmetry. However, the rotation rate $\hat{\theta}$ is different from the point vortex model (3.1). We follow the same procedure as in the inviscid case to obtain this rotation rate: describe the system in an inertial frame originated at the center point $O$, with $x$ axis parallel to the line initially connecting
the vortices. In terms of polar coordinates, the positions of the vortex centers can
be expressed as $z_L = -z_R = -e^{i\theta}/2$. The complex velocity of $z$ is given by

$$\dot{z}_R^* = -i\frac{\dot{\theta} e^{-i\theta}}{2}. \quad (3.3)$$

On the other hand, substituting positions and circulations into (2.29) gives

$$\dot{z}_R^* = \frac{1}{2\pi i e^{i\theta}} \left[ 1 - \exp \left( \frac{-1}{4\tau} \right) \right]. \quad (3.4)$$

Equating these two, the rotation rate based on multi-Gaussian model is given by

$$\dot{\theta} \equiv \frac{d\theta}{dt} = \frac{1}{\pi} \left[ 1 - \exp \left( \frac{-1}{4\tau} \right) \right]. \quad (3.5)$$

Integrating $\dot{\theta}$ in time, one has the orientation angle $\theta$,

$$\theta = \frac{Re}{\pi} \left[ \tau - \exp \left( \frac{-1}{4\tau} \right) + \frac{1}{4} Ei \left( \frac{-1}{4\tau} \right) \right], \quad (3.6)$$

in which the Euler function is defined as $Ei(x) = \int_{-\infty}^{x} \exp(t)/t \, dt$ in principle value’s sense, and it can be numerically calculated up to machine accuracy. Figure 3.2 shows $\dot{\theta}$ and $\theta$ as functions of the scaled time $\tau$. Reynolds number is $Re = 1000$, or equivalently, $\nu = 1/1000$. The same parameter will be used for all cases unless specifically stated otherwise. Initially, $\dot{\theta}(0) = 1/\pi$ is the same as in the inviscid
Figure 3.2: (a) Rotation rate \( \dot{\theta} \) and (b) rotation angle \( \theta \) for the symmetric co-rotating case using multi-Gaussian model. Reynolds number is \( Re = 1000 \).

case given by (3.1), then gradually decays to zero as time goes to infinity \( \tau \to \infty \).

Finally, positions of the two vortex centers are given by

\[
\mathbf{z}_L = -\frac{\cos \theta}{2} - i \frac{\sin \theta}{2}, \quad \mathbf{z}_R = \frac{\cos \theta}{2} + i \frac{\sin \theta}{2}.
\]

(3.7)

Figure 3.3: Symmetric co-rotating pair: (a) Vorticity along the connecting line. Two vorticity peaks decay and merge to a single peak at the origin. (b) Distance between the vorticity peaks and the origin. The peaks merge at time \( \tau = 0.125 \). (c) Vorticity evolution at the origin. The maximum vorticity occurs at \( \tau = 0.0625 \).
Vorticity field  The vorticity field starts as two Dirac delta peaks and eventually converges to a single Gaussian profile, hence the two peaks must merge at the origin at a certain time, though the vortex centers will remain constant distances away from the origin for all time. Vorticity along the connecting line is plotted at various instants in Figure 3.3(a). The distance between one of the vorticity peaks (they are symmetric about the origin) and the origin is denoted by \( b_p \). At the peaks, one has

\[
\frac{\partial \omega}{\partial r} \bigg|_{r=b_p} = 0, \quad \frac{\partial^2 \omega}{\partial r^2} \bigg|_{r=b_p} \leq 0,
\]

where \( r \) is the distance from the origin. The result of \( b_p \) is plotted in Figure 3.3(b). It starts from \( 1/2 \) and reaches 0 at time \( \tau = 0.125 \), which is when the second derivative in the above inequality is equal to 0. Vorticity at the origin is given by

\[
\omega_O = \frac{1}{2\pi \tau} \exp\left(-\frac{1}{16 \tau}\right),
\]

and is plotted in Figure 3.3(c). The maximum vorticity at the origin occurs when \( \tau = 0.0625 \), which is obtained by solving for \( \tau \) such that \( \partial \omega_O / \partial \tau = 0 \).

Absolute velocity field  The absolute velocity field \( \dot{z} \), that is, the velocity field with respect to the inertial frame, can be obtained by substituting (3.7) into (2.28) (which gives the complex velocity \( \dot{z}^* \)), and the streamline evolution in inertial frame is plotted in Figure 3.4. The flow field eventually evolves into a single Gaussian
vortex as expected. While the vortex structure is evolving, it is also rotating unsteadily, and the rotation rate is given by (3.5). It is convenient to express the

\[
Γ
\]

absolute velocity field in a frame co-rotating with the vortex pair at the time dependent rotation rate \( \dot{θ} \). Let \( \xi = (ζ, η) \) denote the position of an arbitrary point expressed in this rotating frame. Transformation from the rotating frame to inertial frame is given by

\[
z = Rξ, \quad R = \begin{bmatrix} \cos θ & -\sin θ \\ \sin θ & \cos θ \end{bmatrix},
\] (3.8)
The positions of the vortex centers in this frame remain \( \xi_L = (-1/2, 0) \) and \( \xi_R = (1/2, 0) \) for all time. The absolute velocity field can be expressed both in the inertial frame, denoted by \( \mathbf{u} = (u_x, u_y) \), or in the rotating frame, denoted by \( \mathbf{v} = (v_\zeta, v_\eta) \). Essentially, \( \mathbf{u} \) and \( \mathbf{v} \) are representing the same velocity field, they are merely expressed in different frames. Transformation between these two is given by

\[
\mathbf{u} = R \mathbf{v} .
\] (3.9)

Mathematically, velocity field \( \mathbf{v} \) can be written in component form,

\[
v_\zeta = -\frac{\eta}{2\pi [(\zeta + 1/2)^2 + \eta^2]} \left\{ 1 - \exp \left[ -\frac{(\zeta + 1/2)^2 + \eta^2}{4\tau} \right] \right\} - \frac{\eta}{2\pi [(\zeta - 1/2)^2 + \eta^2]} \left\{ 1 - \exp \left[ -\frac{(\zeta - 1/2)^2 + \eta^2}{4\tau} \right] \right\},
\]

\[
v_\eta = \frac{\zeta + 1/2}{2\pi [(\zeta + 1/2)^2 + \eta^2]} \left\{ 1 - \exp \left[ -\frac{(\zeta + 1/2)^2 + \eta^2}{4\tau} \right] \right\} + \frac{\zeta - 1/2}{2\pi [(\zeta - 1/2)^2 + \eta^2]} \left\{ 1 - \exp \left[ -\frac{(\zeta - 1/2)^2 + \eta^2}{4\tau} \right] \right\} .
\] (3.10)

Streamline evolution associated with (3.10) is depicted in Figure 3.6 at various instants. Note that the streamline plots in Figure 3.6 are identical to that in Figure 3.4 except the “camera” in Figure 3.6 also rotates with the fluid structure at a rotation rate \( \dot{\theta} \). Among the streamlines, the one passes through the stagnation point (the origin \( O \)) is called the instantaneous separatrix, that is the thick line in Figure 3.6, with arrows showing the directions of the flow. Initially, the separatrix is in \( \infty \)-shape centered at \( O \), and the flow field is divided into three regions by the
Figure 3.6: Streamlines of absolute velocity field for the symmetric co-rotating pair in rotating frame at various instants $\tau = 0, 0.025, 0.05$ and $0.075$. Separatrices are depicted in thick lines with arrows representing directions of the flow. Each plot shows a $[-1,1] \times [-1,1]$ window in the $(\zeta, \eta)$ plane.

separatrix: the two finite regions around the vortex centers, and an infinite region that consists of the rest of the flow field. As time evolves, the separatrix shrinks towards $O$, and the two finite regions also shrink accordingly. Eventually, at a finite bifurcation time $\tau_b$, the separatrix collapses at $O$, only the infinite region remains in the flow field.

Several critical issues need to be pointed out: first, contrary to a steady state flow field, a time dependent separatrix does not constitute barriers to fluid motion and fluid particles typically move across this separatrix. This can be understood from the shrinking of the two finite regions: since this is an incompressible flow, fluid particles have no place to go but to “leak” across the separatrix, and this “leakage” typically occurs along the unstable manifolds associated with the hyperbolic points. Second, since the governing equation of the velocity field is implicitly time dependent via $\theta$, typically one can only look for the instantaneous stagnation points in the flow field. To this end, time $\tau$ should be treated as a parameter of the system. In another word, at any instant $\tau$, one should take a “snapshot” of the whole flow.
field, then analyze the stagnation points (also called the \textit{fixed points}) within this snapshot. The sequence of these snapshots represents the complete evolution of the flow field. The positions of the instantaneous stagnation points typically do not remain fixed in a time dependent flow field. With this understanding, for simplicity, we still refer to the instantaneous separatrices and instantaneous stagnation points as separatrices and fixed points, respectively.

The positions of the fixed points can be obtained by setting the right-hand-sides of (3.10) to zero. Due to symmetry, the fixed points must lie on the $\zeta$ and/or $\eta$ axes. On the $\eta$ axis, we have $\zeta = 0$, the non-trivial governing equation for the fixed points is given by

$$
-\frac{2\eta}{1/4 + \eta^2} \left[ 1 - \exp\left( -\frac{1/4 + \eta^2}{4\tau} \right) \right] = 0.
$$

The solution is $\eta = 0$. Therefore, one solution of the fixed point is the origin $(0, 0)$. On the $\zeta$ axis ($\eta = 0$), the non-trivial governing equation for fixed points is given by

$$
\frac{1}{\zeta + 1/2} \left\{ 1 - \exp\left[ -\frac{(\zeta + 1/2)^2}{4\tau} \right] \right\} + \frac{1}{\zeta - 1/2} \left\{ 1 - \exp\left[ -\frac{(\zeta - 1/2)^2}{4\tau} \right] \right\} = 0.
$$

One finds a pair of solutions denoted by $\pm \zeta_f$, which can not be expressed in regular form. The result is plotted in Figure 3.7. This figure should be understood as follows: for any time $\tau$, a “snapshot” of the flow field is taken, and Figure 3.7 shows the position of this pair in every snapshot. The pair $(\pm \zeta_f, 0)$ initially locates
at \((\pm 1/2, 0)\) (the positions of the vortex centers), then collapses at the origin at a finite time \(\tau_b \approx 0.05\), which is a bifurcation time of the system. Hence, before \(\tau_b\), the system has 3 fixed points \((0, 0)\) and \((\pm \zeta_f, 0)\); after \(\tau_b\), the system only has one fixed point \((0, 0)\). Interestingly, this bifurcation time does not coincide with the time the two vorticity peaks merges at the origin, which is \(\tau = 0.125\). And, obviously, voriticity contours do not coincide with streamlines.

Figure 3.7: Horizontal components of the fixed points \((\pm \zeta_f, 0)\). They collapse at the origin at a finite bifurcation time \(\tau_b \approx 0.05\).

\[
\begin{array}{c}
\text{(a) } \tau = 0 \\
\text{(b) } \tau = 0.1 \\
\text{(c) } \tau = 0.125 \\
\text{(d) } \tau = 0.175
\end{array}
\]

Figure 3.8: Streamlines associated with the relative velocity field for the symmetric co-rotating pair in the rotating frame at various instants \(\tau = 0, 0.1, 0.125\) and 0.175. Each plot shows a \([-1.5, 1.5] \times [-1.5, 1.5]\) window in the \((\zeta, \eta)\) plane.
Relative velocity field  As will be shown in the passive particle evolution, the sequence of the absolute velocity field evolution does not give enough insights of the flow field, it is the relative velocity field that reveals the most important features of the system. A relative velocity field is the field observed with respect to the rotating frame. Naturally, the relative velocity field is also expressed in the rotating frame. Mathematically, the governing equation for the relative velocity field is obtained by taking time derivative of the transformation (3.8),

\[ \dot{z} = \dot{R}_\xi + R \dot{\xi}, \]

Hence,

\[ \dot{\xi} = R^T \dot{z} - R^T \dot{R} R^T z = v - \dot{\theta} \xi^\perp, \quad (3.11) \]

where \( v \) is the absolute velocity field expressed in the rotating frame given in (3.10), and \( \xi^\perp = (-\eta, \zeta) \). In another word, the relative velocity field is obtained by subtracting a rigid body rotation with the rotation rate \( \dot{\theta} \) from the rotated absolute velocity field \( v \). Physically, this relation can be understood as follows: as time evolves, the vorticity field, initially concentrated at \( z_L(0) \) and \( z_R(0) \), begins to spread spatially and induce a velocity field similar to that of a Rankine vortex with a time dependent core. The reader is reminded that for a Rankine vortex located at the origin with circulation \( 2\Gamma \) (corresponding to the total circulation of the pair),
the radial velocity $v_r$ at an arbitrary point $(\zeta, \eta)$ is zero, and the tangential velocity $v_\theta$ is given by

$$v_\theta(r) = \begin{cases} \frac{\Gamma}{\pi R_c^2} r, & \text{for } r \leq R_c \\ \frac{\Gamma}{\pi} \frac{1}{r}, & \text{for } r > R_c \end{cases} \quad (r^2 = \zeta^2 + \eta^2). \quad (3.12)$$

The value $R_c$ is referred to as the core of the Rankine vortex. For $r \leq R_c$, the velocity field corresponds to a rigid rotation, while for $r > R_c$, the velocity decays proportionally to the inverse of the distance $r$. The velocity profile of the Rankine vortex (dashed line) is superimposed on the tangential component of the absolute velocity induced by the viscously evolving co-rotating pair along the connecting line (solid line) at three instants in Figure 3.9. Closer to the origin, the velocity field of the pair is similar to a rigid rotation, and the rotation rate is given by $\dot{\theta}$ in (3.5). As $r$ increases beyond $R_c$, the velocity field of the pair decays similarly to an inverse decay with vorticity $2\Gamma$. Since the rotation rate is time dependent, the core size $R_c$

Figure 3.9: Velocity field induced by the symmetric co-rotating pair is similar to that of a Rankine vortex. See text for details.
of the Rankine vortex, obtained by \(2\Gamma/2\pi R_c^2 = \hat{\theta}\), is also time dependent, and it increases with time as shown in Figures 3.9. Much of the interesting transitional dynamics occurs close to the origin, typically within a circular area of radius 1. Note that (most of the time) \(R_c\) is larger than 1, hence the rigid rotation region covers our interested area. Although the difference between the solid and dashed lines is small (especially close to the origin), it is this small difference that reveals the most important information of the flow field.

Motivated by the analogy with the Rankine vortex, we now examine the evolution of the relative velocity field. In component form, (3.11) can be expressed as

\[
\dot{\zeta} = \hat{\theta}\eta - \frac{\eta}{2\pi[(\zeta + 1/2)^2 + \eta^2]} \left\{ 1 - \exp \left[ -\frac{(\zeta + 1/2)^2 + \eta^2}{4\tau} \right] \right\},
\]

\[
\dot{\eta} = -\hat{\theta}\zeta + \frac{\zeta + 1/2}{2\pi[(\zeta + 1/2)^2 + \eta^2]} \left\{ 1 - \exp \left[ -\frac{(\zeta + 1/2)^2 + \eta^2}{4\tau} \right] \right\},
\]

Similar to the procedure in the absolute velocity field, we look for fixed points of the system by setting the right-hand-sides of above equations to zero. Again, fixed points must lie on the \(\zeta\) and/or \(\eta\) axis. On the \(\eta\) axis, the governing equation for \(\eta\) component of the fixed point is given by

\[
\hat{\theta}\eta - \frac{\eta}{\pi(1/4 + \eta^2)} \left[ 1 - \exp \left( -\frac{1/4 + \eta^2}{4\tau} \right) \right] = 0.
\]
The fixed points on the $\eta$ axis are $(0, 0)$ and $(0, \pm \sqrt{3}/2)$. On the $\zeta$ axis, the governing equation for $\zeta$ component of the fixed point is

$$-\dot{\zeta} + \frac{1}{\pi(\zeta + 1/2)} \left\{ 1 - \exp \left[ -\frac{(\zeta + 1/2)^2}{4\tau} \right] \right\} + \frac{1}{\pi(\zeta - 1/2)} \left\{ 1 - \exp \left[ -\frac{(\zeta - 1/2)^2}{4\tau} \right] \right\} = 0.$$  

The solutions are $(0, 0)$, $(\pm 1/2, 0)$ and another pair $(\pm \zeta_f^*, 0)$ that needs to be numerically solved (the superscript $()^*$ is to distinguish the fixed point for the relative velocity field from the absolute field). Figure 3.10 shows the horizontal components of this pair as functions of time. One can see the pair starts from $(\pm \sqrt{5}/2, 0)$ (which is obtained by setting $\tau = 0$ in the equation) then coincides with $(\pm 1/2, 0)$ at a finite time $\tau_2^* \approx 0.118$, which is the a bifurcation time of the topological change of the separatrices in the relative field. Eventually, the pair collapses at the origin at $\tau_4^* \approx 0.132$, which is another bifurcation time. As will be discussed later, there are 4 bifurcation times in total. Before, $\tau_4^*$, the system has 7 fixed points:
(0, 0), (±1/2, 0), (±ζ^*, 0) and (0, ±√3/2); after τ_1^*, the system has 5 fixed points: (0, 0), (±1/2, 0) and (0, ±√3/2).

The hyperbolic and/or elliptic characters of these fixed points can be analyzed by linearizing (3.11) about the fixed points and studying the eigenvalue problems associated with the linearized system: the eigenvalues are always in pair. If the eigenvalues are pure imaginary, then the corresponding fixed point is an elliptic point. If the eigenvalues are real, they are always equal and opposite, therefore one of the eigenvalues is positive, hence the fixed point is a hyperbolic point. Note this linearization is in space only, time is treated as a bifurcation parameter. First, we linearize the system around the origin: assume ζ = 0 + δζ and η = 0 + δη, where δ indicates an infinitesimal perturbation, the velocity is given by ˙ζ = δ ˙ζ, ˙η = δ ˙η.

Substituting into the velocity equation (3.11) gives

\[
\delta ˙\zeta = \dot{\theta} \delta \eta - \frac{\delta \eta}{2\pi(1/4 + \delta \zeta + \delta \zeta^2 + \delta \eta^2)} \left[ 1 - \exp \left( -\frac{1/4 + \delta \zeta + \delta \zeta^2 + \delta \eta^2}{4\tau} \right) \right]
\]

\[
- \frac{\delta \eta}{2\pi(1/4 - \delta \zeta + \delta \zeta^2 + \delta \eta^2)} \left[ 1 - \exp \left( -\frac{1/4 - \delta \zeta + \delta \zeta^2 + \delta \eta^2}{4\tau} \right) \right]
\]

\[
= \dot{\theta} \delta \eta - \frac{2\delta \eta}{\pi} \left[ 1 - 4\delta \zeta + O(\delta \zeta^2, \delta \eta^2) \right] \left[ 1 - \exp \left( -\frac{1}{16\tau} \right) \left( 1 - \frac{\delta \zeta}{4\tau} \right) + O(\delta \zeta^2, \delta \eta^2) \right]
\]

\[
- \frac{2\delta \eta}{\pi} \left[ 1 + 4\delta \zeta + O(\delta \zeta^2, \delta \eta^2) \right] \left[ 1 - \exp \left( -\frac{1}{16\tau} \right) \left( 1 + \frac{\delta \zeta}{4\tau} \right) + O(\delta \zeta^2, \delta \eta^2) \right]
\]

\[
\approx \delta \eta \left\{ \dot{\theta} - \frac{4}{\pi} \left[ 1 - \exp \left( -\frac{1}{16\tau} \right) \right] \right\},
\]
\[ \delta \dot{\eta} = -\dot{\theta} \delta \zeta + \frac{\delta \zeta + 1/2}{2\pi (1/4 + \delta \zeta + \delta \zeta^2 + \delta \eta^2)} \left[ 1 - \exp \left( -\frac{1/4 + \delta \zeta + \delta \zeta^2 + \delta \eta^2}{4\tau} \right) \right] \]

\[ + \frac{\delta \zeta - 1/2}{2\pi (1/4 - \delta \zeta + \delta \zeta^2 + \delta \eta^2)} \left[ 1 - \exp \left( -\frac{1/4 - \delta \zeta + \delta \zeta^2 + \delta \eta^2}{4\tau} \right) \right] \]

\[ = -\dot{\theta} \delta \zeta \]

\[ + \frac{2\delta \zeta + 1}{\pi} \left[ 1 - 4\delta \zeta + O(\delta \zeta^2, \delta \eta^2) \right] \left[ 1 - \exp \left( \frac{-1}{16\tau} \right) \left( 1 - \frac{\delta \zeta}{4\tau} \right) + O(\delta \zeta^2, \delta \eta^2) \right] \]

\[ + \frac{2\delta \zeta - 1}{\pi} \left[ 1 + 4\delta \zeta + O(\delta \zeta^2, \delta \eta^2) \right] \left[ 1 - \exp \left( \frac{-1}{16\tau} \right) \left( 1 + \frac{\delta \zeta}{4\tau} \right) + O(\delta \zeta^2, \delta \eta^2) \right] \]

\[ \approx \delta \zeta \left\{ -\dot{\theta} + \frac{1}{2\pi \tau} \exp \left( \frac{-1}{16\tau} \right) - \frac{4}{\pi} \left[ 1 - \exp \left( \frac{-1}{16\tau} \right) \right] \right\}, \]

where \( O(x) \) indicates the same and higher order terms of \( x \). The last steps are approximations obtained by neglecting the nonlinear terms. Therefore, the linearized equation around the origin can be written in matrix form

\[ \begin{pmatrix} \delta \dot{\zeta} \\ \delta \dot{\eta} \end{pmatrix} = \begin{pmatrix} 0 & C_{12} \\ C_{21} & 0 \end{pmatrix} \begin{pmatrix} \delta \zeta \\ \delta \eta \end{pmatrix}, \quad (3.13) \]

where

\[ C_{12} = \dot{\theta} - \frac{4}{\pi} \left[ 1 - \exp \left( \frac{-1}{16\tau} \right) \right], \]

\[ C_{21} = -\dot{\theta} + \frac{1}{2\pi \tau} \exp \left( \frac{-1}{16\tau} \right) - \frac{4}{\pi} \left[ 1 - \exp \left( \frac{-1}{16\tau} \right) \right]. \]
Eigenvalues of (3.13) are given by

$$\lambda_{1,2} = \pm \sqrt{C_{12}C_{21}} .$$  \hspace{1cm} (3.14)

Real and imaginary parts of this pair of eigenvalues are shown in Figure 3.11. Initially the origin is a hyperbolic point, after bifurcation time $\tau_4^*$, it becomes an elliptic point.

![Figure 3.11: Real and imaginary parts of the eigenvalues associated with (0, 0) for the symmetric co-rotating pair. Before $\tau_4^*$, the origin is a hyperbolic point, after $\tau_4^*$, it becomes an elliptic point.](image)

Next, we linearize (3.11) around (1/2, 0), which gives the same result as (-1/2, 0).

Assume $\zeta = 1/2 + \delta\zeta$ and $\eta = 0 + \delta\eta$, hence $\dot{\zeta} = \delta\dot{\zeta}$ and $\dot{\eta} = \delta\dot{\eta}$. Substituting these perturbations and (3.5) into (3.11), the linearized equations around (1/2, 0) can be written in the same matrix form as (3.13), but $C_{12}$ and $C_{21}$ are now given by

$$C_{12} = \dot{\theta} - \frac{1}{8 \pi \tau} - \frac{1}{2 \pi} \left[ 1 - \exp \left( \frac{-1}{4 \tau} \right) \right] ,$$

$$C_{21} = -\frac{3}{2 \pi} \left[ 1 - \exp \left( \frac{-1}{4 \tau} \right) \right] + \frac{1}{8 \pi \tau} \left[ 1 + 2 \exp \left( \frac{-1}{4 \tau} \right) \right] .$$
Real and imaginary parts of these eigenvalues are plotted in Figure 3.12. Initially, the pair $(\pm 1/2,0)$ are elliptic points, after the bifurcation time $\tau_2^*$, they become hyperbolic points.

Next, we linearize (3.11) around $(0, \sqrt{3}/2)$, which gives the same result as $(0, -\sqrt{3}/2)$. Assume $\zeta = 0 + \delta \zeta$ and $\eta = \sqrt{3}/2 + \delta \eta$, hence $\dot{\zeta} = \delta \dot{\zeta}$ and $\dot{\eta} = \delta \dot{\eta}$. The linearized equations around $(0, \sqrt{3}/2)$ can also be written in the same matrix form as (3.13), but $C_{12}$ and $C_{21}$ are now given by

$$C_{12} = -\frac{3}{8\pi \tau} \exp\left(-\frac{1}{4\tau}\right) + \frac{3}{2\pi} \left[1 - \exp\left(-\frac{1}{4\tau}\right)\right],$$

$$C_{21} = -\dot{\theta} + \frac{1}{8\pi \tau} \exp\left(-\frac{1}{4\tau}\right) + \frac{1}{2\pi} \left[1 - \exp\left(-\frac{1}{4\tau}\right)\right].$$

Real and imaginary parts of the eigenvalues are plotted in Figure 3.13. One can see that the pair $(0, \pm \sqrt{3}/2)$ are always elliptic points.
Finally, we linearize (3.11) around the moving fixed points \((\pm \zeta_f^*, 0)\). Following similar procedure as before, the linearized equations can be written in the same matrix form as (3.13), but \(C_{12}\) and \(C_{21}\) are now given by

\[
C_{12} = \dot{\theta} - \frac{1}{2\pi (\zeta_f^* + 1/2)^2} \left\{ 1 - \exp \left[ -\frac{\left(\zeta_f^* + 1/2\right)^2}{4\tau} \right] \right\} \\
- \frac{1}{2\pi (\zeta_f^* - 1/2)^2} \left\{ 1 - \exp \left[ -\frac{\left(\zeta_f^* - 1/2\right)^2}{4\tau} \right] \right\},
\]

\[
C_{21} = -\dot{\theta} + \frac{1}{4\pi \tau} \left\{ \exp \left[ -\frac{\left(\zeta_f^* + 1/2\right)^2}{4\tau} \right] + \exp \left[ -\frac{\left(\zeta_f^* - 1/2\right)^2}{4\tau} \right] \right\} \\
- \frac{1}{4\pi (\zeta_f^* + 1/2)^2} \left\{ 1 - \exp \left[ -\frac{\left(\zeta_f^* + 1/2\right)^2}{4\tau} \right] \right\} \\
- \frac{1}{4\pi (\zeta_f^* - 1/2)^2} \left\{ 1 - \exp \left[ -\frac{\left(\zeta_f^* - 1/2\right)^2}{4\tau} \right] \right\}.
\]

Real and imaginary parts of the eigenvalues associated with \((\pm \zeta_f^*, 0)\) are shown in Figure 3.14. One can see that \((\pm \zeta_f^*, 0)\) are hyperbolic points before \(\tau_2^*\); at \(\tau_2^*\), this pair coincides with \((\pm 1/2, 0)\); after \(\tau_2^*\) and before \(\tau_4^*\), they are elliptic points; at \(\tau_4^*\),

Figure 3.13: Real and imaginary parts of the eigenvalues associated with \((0, \pm \sqrt{3}/2)\) for the symmetric co-rotating pair. This pair is always elliptic points.
they collapse at origin; after $\tau^*_4$, this pair remains elliptic points. Indeed, the part in Figure 3.14 after $\tau^*_4$ is exactly the same with the part in Figure 3.11 after $\tau^*_4$.

We have shown the fixed points and the associated eigenvalues for the relative velocity field of the symmetric co-rotating pair in rotating frame. Notice that all

![Figure 3.14: Real and imaginary parts of the eigenvalues of $(\pm \zeta^*_f, 0)$ for the symmetric co-rotating pair. Before $\tau^*_2$, this pair is hyperbolic points. After $\tau^*_2$, they become elliptic points](image)

![Figure 3.15: Evolution of fixed points and separatrices associated with the relative velocity field for the symmetric co-rotating pair in rotating frame. The arrows on separatrices show velocity directions. Hyperbolic points are represented by cross-sections of separatrices. Elliptic points are small circles.](image)
eigenvalues eventually become zero, because the velocity field eventually diffuses to zero everywhere due to viscosity. One can plot the instantaneous streamlines based on the velocity field given by (3.11). Figure 3.8 shows the streamlines at various time in rotating frame. Specifically, among the streamlines, the ones pass through the hyperbolic points are separatrices. Figure 3.15 shows the fixed points and separatrices at eight instants. At $\tau = 0$, Figure 3.15(a) shows that the separatrices consist of inner, middle and outer separatrices: the inner separatrix bounds fluid regions around $(\pm 1/2, 0)$ and crosses itself at the hyperbolic point $(0, 0)$; the middle and outer separatrices bound the fluid regions around $(0, \pm \sqrt{3}/2)$ and cross themselves at the hyperbolic pair $(\pm \zeta^*_f, 0)$. A band exists between the inner and middle separatrices. As soon as $\tau > 0$, this band starts to shrink. At time $\tau_1^* \approx 0.068$, this band shrinks to zero because the inner and middle separatrices collapse onto each other, see Figure 3.15(b). This is the first bifurcation time in terms of topological change of the separatrices. The second bifurcation $\tau_2^* \approx 0.118$ is depicted in Figure 3.15(d). This bifurcation occurs when the hyperbolic points at $(\pm \zeta^*_f, 0)$ coincide with the elliptic points $(\pm 1/2, 0)$, causing them to become hyperbolic points. The third bifurcation takes place at $\tau_3^* \approx 0.127$ due to yet another collapse of the separatrices associated with the now elliptic points at $(\pm \zeta^*_f, 0)$ and separatrices associated with the elliptic pair $(0, \pm \sqrt{3}/2)$, see Figure 3.15(f). The fourth bifurcation occurs at $\tau_4^* \approx 0.132$ when the pair $(\pm \zeta^*_f, 0)$ collapses at the origin, turning it into an elliptic point, see Figure 3.15(h). After $\tau_4^*$, the topology of the separatrices remains the same.
Figure 3.16: (a) Dye visualization of a merging symmetric co-rotating vortex pair from Trieling et al. [101]. (b) Vorticity contours of the symmetric merger from Navier-Stokes simulation in Brandt & Nomura [16].

**Passive particle evolution** The experimental result of a symmetric merger from Trieling et al. [101] is shown again in Figure 3.16(a). The authors used two different chemical compounds as the orange and green colored dyes to visualize the merging process. These dyes are essentially passive particles being advected by the local velocity field. Vorticity contours based on the results of high fidelity Navier-Stokes simulation from Brandt & Nomura [16] is shown in Figure 3.16(b). Clearly, one can see remarkable resemblance between the two. As will be argued in Section 3.4, the vorticity contour based on multi-Gaussian model does not capture the Navier-Stokes solution, as hinted by the vorticity evolution on the connecting line shown in Figure 3.3(a). In particular, the vorticity evolution of the model does not capture the filaments formed in the outside. However, inspired by the dye visualization, we also seed passive particles in the flow field, and let them be advected by the local
velocity field, i.e. the governing equation for the position of each particle is given by \((2.28)\). For convenience, we will rotate the passive particle evolution into the rotating frame following the particle transformation \((3.8)\). Since at time \(\tau = 0\) the vorticity is concentrated at two points, we shall let the vortex cores diffuse for a short amount of time \(\tau_0 = 0.003\), then seed the flow with passive particles of six different colors as shown in Figure 3.17 to distinguish particles within the left (red) and right (blue) cores of radii \(a_0 = \sqrt{4\tau_0} \approx 0.110\), left (yellow) and right (teal) rings of inner radii \(a_0\) and outer radii \(2a_0\), and left (purple) and right (green) rings of inner radii \(2a_0\) and outer radii \(3a_0\). Since \(a_0\) is the core, i.e. the standard deviation of a Gaussian function, the \(3a_0\) range accounts for more than 99% of the total initial vorticity distribution, known as the 3-sigma rule.

We overlay the separatrices in rotating frame on top of the snapshots of particle evolution. The separatrices are very good indications of the particle evolution. Overall, Figure 3.17 resembles the experimental work or numerical simulation very well, especially the formation of filaments. The initial state is shown in Figure 3.17(a). As time evolves, the particles first rotate around the two elliptic points coincide with the vortex centers. Before \(\tau = 0.0235\), they are confined within the inner separatrix (the region bounded by the inner separatrix is referred to as the inner cores), see Figure 3.17(b). After \(\tau = 0.0235\), some of the outer particles (purple and green) start to “leak” into the band between inner separatrix and middle separatrix (referred to as the exchange band), and they mix while circulate in the band around the inner cores. As the exchange band continues to shrink, some
of the outer particles will consequently leak outside the exchange band when they encounter the hyperbolic points (along the unstable manifolds associated with the hyperbolic points). At the first bifurcation time $\tau_1^* \approx 0.068$, the exchange band disappears, the inner particles still remain in the inner cores, some of the outer particles are in the inner cores while the rest of the outer particles now erode into

Figure 3.17: Passive particle evolution for the symmetric co-rotating pair.
the filaments in the regions bounded by the outer separatrix and the middle separatrix (referred to as the outer recirculation area), see Figure 3.17(d). Since the separatrices are time dependent Eulerian streamlines, the passive particles will not faithfully follow the separatrices. Therefore, when the particles travel closely along the outer separatrix and recirculate close to the opposite hyperbolic point, they will “escape” to the region outside the outer separatrix (referred to as the outside area), see Figure 3.17(e). In the mean time, some of the middle (yellow and teal) (and later inner (red and blue) particles) encounter the hyperbolic points and enter the outer recirculation area, they mix with the outer particles while traveling close to the outer separatrix. At the second bifurcation time $\tau_2^* \approx 0.118$, the moving hyperbolic points collide with the elliptic points, and most of the inner particles still remain close to the now hyperbolic points $(\pm 1/2, 0)$, see Figure 3.17(f). As the separatrices continue to evolve, the particles continue to mix in the outside area, but closer to the origin the six colored particles are separated by clear boundaries. After the last bifurcation time $\tau_4^* \approx 0.132$, the topology of the separatrices remains the same, and the shape of the streamlines will not vary much (although the magnitude of the velocity will continue to decay). Hence, after $\tau_4^* \approx 0.132$, the particles confined within the inner separatrix (which include all six colors) will mostly remain inside, and the rest of the particles will circulate in the outside area around the outer separatrix. As shown in Figure 3.17(h) and (i), due to the difference in the magnitude of velocities inside the inner separatrix, the boundaries between
the inside six colored particles will start to stretch, causing the particles to mix in layers.

**Summary** The sequences of the bifurcation changes of the separatrices in absolute velocity field and relative velocity field are summarized in Figure 3.18 and 3.19, respectively, which are visually different but topologically equivalent to, i.e. homotopic equivalences of the original separatrices. The bifurcation states are bounded by boxes. The timeline of all important events for the symmetric co-rotating pair based on the multi-Gaussian model is plotted in Figure 3.20.
3.2 Asymmetric co-rotating vortex pair

In inviscid flow, an asymmetric co-rotating pair consists of two point vortices of the same sign of vorticity, but the strengths are different, and the net circulation is not zero. The positions of the left and right vortices in complex plane are $z_L$ and $z_R$, respectively, and they are separated by distance $b_0$. The strengths of the vortices are $\Gamma_L$ and $\Gamma_R$. One can describe the motion in an inertial frame with $x$ axis parallel to the line connecting the two vortices, and originated between the
vortices, such that if the distance between \( \mathbf{z}_L (\mathbf{z}_R) \) and the origin is denoted as \( r_L \) (\( r_R \)), one has the following relation:

\[
\frac{\Gamma_L}{\Gamma_R} = \frac{r_R}{r_L} \quad \Rightarrow \quad r_L = \frac{\Gamma_R}{\Gamma_L + \Gamma_R} b_0, \quad r_R = \frac{\Gamma_L}{\Gamma_L + \Gamma_R} b_0.
\]

The rotation rate of the structure is given by

\[
\dot{\theta} = \frac{\Gamma_L + \Gamma_R}{2\pi b_0^2}.
\]

In this work, for concreteness, we will discuss the example that has dimensionless strengths \( \Gamma_L = 4/3, \Gamma_R = 2/3 \) and distance 1. Therefore, one has the following

\[
r_L = \frac{1}{3}, \quad r_R = \frac{2}{3}, \quad \dot{\theta} = \frac{1}{\pi}.
\] (3.15)

**Dynamics of vortex centers**  Now we consider the problem in viscous fluid using the multi-Gaussian model. The vorticity field is given by

\[
\mathbf{\omega} = \frac{1}{6\pi\tau} \left[ 2 \exp \left( -\frac{\| \mathbf{z} - \mathbf{z}_L \|^2}{4\tau} \right) + \exp \left( -\frac{\| \mathbf{z} - \mathbf{z}_R \|^2}{4\tau} \right) \right].
\]

Similar to the symmetric case, the induced velocities of the vortex centers are always perpendicular to their connecting line, therefore the distances between the vortex centers and the origin are constants, and the system rotates around the origin. The
rotation rate $\dot{\theta}$ can be obtained similar as before by focusing on the complex velocity of $z_R$,

$$
\dot{z}_R^* = -i \frac{2}{3} \dot{\theta} e^{-i\theta} = \frac{4/3}{2 \pi i e^{i\theta}} \left[ 1 - \exp \left( \frac{-1}{4 \tau} \right) \right].
$$

Therefore, the rotation rate and rotation angle are given by

$$
\dot{\theta} = \frac{1}{\pi} \left[ 1 - \exp \left( \frac{-1}{4 \tau} \right) \right], \quad \theta = \frac{\text{Re}}{\pi} \left[ \tau - \exp \left( \frac{-1}{4 \tau} \right) \tau + \frac{1}{4} \text{Ei} \left( \frac{-1}{4 \tau} \right) \right],
$$

which are identical to the expressions in (3.5) and (3.6) obtained in the symmetric case. One can verify that the rotation rate (therefore the rotation angle) of the co-rotating pair only depends on the total circulation in the flow and initial separation between the two vortex centers, but not on the strength ratio $\Gamma_L/\Gamma_R$ (which is 2 in this case). The positions of the two vortices are given by

$$
z_L = -\frac{\cos \theta}{3} - i \frac{\sin \theta}{3}, \quad z_R = \frac{2 \cos \theta}{3} + i \frac{2 \sin \theta}{3}.
$$

(3.16)

**Vorticity field** Similar to the symmetric case, vorticity of the asymmetric pair also starts with two Dirac delta peaks and eventually converges to a single Gaussian profile. Vorticity along the line that connects the vortex centers is plotted at various instants in Figure 3.22(a). Distances between the vorticity peaks and the origin are denoted $b_L$ and $b_R$ for the respective (initial) left and right vortices. They are obtained by solving $\partial \omega / \partial r = 0$, and $\partial^2 \omega / \partial r^2 \leq 0$, $r$ being the distance from the origin. Unlike the symmetric case, the distance between the vorticity valley and the
Figure 3.22: Asymmetric case: (a) Vorticity along the connecting line. One vorticity peak (the right one) eventually decays and merges with the valley. (b) Distance between the vorticity peaks and valley and the origin as functions of time. One peak and the valley merge when $\tau \approx 0.0725$ at $b_{\text{vanish}} \approx 0.491$. (c) Vorticity evolution at the origin. The maximum vorticity occurs at $\tau \approx 0.0304$.

origin is not 0 for the asymmetric case, it is denoted as $b_V$, and obtained by solving $\partial \omega / \partial r = 0$, but $\partial^2 \omega / \partial r^2 \geq 0$. The result is plotted in Figure 3.22(b). The right peak and valley collapse when $\tau \approx 0.0725$ at $b_{\text{vanish}} \approx 0.491$, and annihilate each other. After $\tau \approx 0.0725$, the left peak remains to be the single peak in the field. Vorticity at the origin is given by

$$\omega_O = \frac{1}{6\pi \tau} \left[ 2 \exp \left( -\frac{1}{36\tau} \right) + \exp \left( -\frac{1}{9\tau} \right) \right],$$

and it is plotted in Figure 3.22(c). The maximum vorticity at the origin occurs at $\tau \approx 0.0304$.

**Absolute velocity field** The streamlines of the asymmetric co-rotating pair in inertial frame are depicted in Figure 3.23. Similar to the symmetric case, the flow field eventually evolves into a single Gaussian vortex centered at origin. While the structure is evolving, it is also rotating unsteadily, and the rotation rate $\dot{\theta}$ is given
by (3.5). We also express the absolute velocity field in a frame $\xi = (\zeta, \eta)$ co-rotating with the structure at the time dependent rotation rate. The positions of the vortex centers in this frame remain $\xi_L = (-r_L, 0) = (-1/3, 0)$ and $\xi_R = (r_R, 0) = (2/3, 0)$ for all time. The absolute velocity with respect to the inertial frame but expressed in the rotating frame $v = (v_\zeta, v_\eta)$ can be given in component form.

Figure 3.23: Streamlines of absolute velocity field for the asymmetric co-rotating pair in inertial frame at $\tau = 0, 0.008, 0.0146$ and $0.05$. Each plot shows a $[-1, 1] \times [-1, 1]$ window in the $(x, y)$ plane.

Figure 3.24: Streamlines of absolute velocity field for the asymmetric co-rotating pair in rotating frame at $\tau = 0, 0.008, 0.0146$ and $0.05$. Each plot shows a $[-1, 1] \times [-1, 1]$ window in the $(\zeta, \eta)$ plane.
\[ v_\zeta = -\frac{2\eta}{6\pi[(\zeta + 1/3)^2 + \eta^2]} \left\{ 1 - \exp \left[ -\frac{(\zeta + 1/3)^2 + \eta^2}{4\tau} \right] \right\} \]
\[ \quad - \frac{\eta}{6\pi[(\zeta - 1/3)^2 + \eta^2]} \left\{ 1 - \exp \left[ -\frac{(\zeta - 2/3)^2 + \eta^2}{4\tau} \right] \right\}, \]

\[ v_\eta = \frac{2(\zeta + 1/3)}{6\pi[(\zeta + 1/3)^2 + \eta^2]} \left\{ 1 - \exp \left[ -\frac{(\zeta + 1/3)^2 + \eta^2}{4\tau} \right] \right\} \]
\[ \quad + \frac{\zeta - 1/3}{6\pi[(\zeta - 2/3)^2 + \eta^2]} \left\{ 1 - \exp \left[ -\frac{(\zeta - 2/3)^2 + \eta^2}{4\tau} \right] \right\}. \]

The streamlines associated with (3.17) are plotted in Figure 3.24. The separatrices are depicted in thick lines with arrows representing the directions of the flow. Fixed points of the velocity field are obtained by setting the right-hand sides of (3.17) to zero. Due to symmetry about the \( \zeta \) axis in the velocity field, the fixed points must lie on the \( \zeta \) axis, but since the flow field is not symmetric about the \( \eta \) axis, one should not anticipate fixed points on the \( \eta \) axis. A total of 3 fixed points are found as shown in Figure 3.24(a), denoted by \((\zeta_{f1}, 0)\), \((\zeta_{f2}, 0)\) and \((\zeta_{f3}, 0)\). The horizontal components of these fixed points are numerically solved, and the results are shown in Figure 3.25. At time \( \tau = 0 \), the streamlines are identical to the inviscid case, with a separatrix surrounding the two vortex centers \((-r_L, 0)\) and \((r_R, 0)\), which coincide with the positions of the elliptic points \((\zeta_{f1}(0), 0)\) and \((\zeta_{f3}(0), 0)\), and crosses at the hyperbolic point \((\zeta_{f2}(0), 0)\), as shown in Figure 3.24(a). As time evolves, positions of the fixed points change as shown in Figure 3.25, and the separatrix evolves accordingly. For all time, \((\zeta_{f1}, 0)\) remains an elliptic point, which starts from \((-1/3, 0)\) and approaches the origin as \( \tau \rightarrow \infty \). The other two fixed points, however, only exist when \( \tau \leq \tau_b \approx 0.0146 \). They start from \((\zeta_{f2}(0) = 1/3, 0)\)
Figure 3.25: Horizontal components of fixed point in absolute velocity field for the asymmetric co-rotating pair in the rotating frame. (a) \((\zeta_{f1}, 0)\), it approaches \((0, 0)\) as \(\tau \to \infty\), (b) \((\zeta_{f2}, 0)\) and \((\zeta_{f3}, 0)\), they collapse at bifurcation time \(\tau_b \approx 0.0146\) at the location \((\zeta_{\text{vanish}} \approx 0.447, 0)\). After \(\tau_b\), \((\zeta_{f2}, 0)\) and \((\zeta_{f3}, 0)\) vanish, only \((\zeta_{f1}, 0)\) remains in the flow field.

and \((\zeta_{f3}(0) = 2/3, 0)\), respectively, and the right ring in the asymmetric \(\infty\)-shaped separatrix shrinks as they approach, see Figure 3.24(b). Eventually they collapse at the location \((\zeta_{\text{vanish}} \approx 0.447, 0)\) at bifurcation time \(\tau_b\), and the right ring vanishes completely. After \(\tau_b\), only \((\zeta_{f1}, 0)\) remains in the flow field. Note that the flow field converges to a single Gaussian vortex centered at the origin as anticipated.

Figure 3.26: Streamlines associated with the relative velocity field for the asymmetric co-rotating pair in rotating frame at various instants \(\tau = 0, 0.05, 0.1\) and \(0.2\). Each plot shows a \([-1.5, 1.5] \times [-1.5, 1.5]\) window in the \((\zeta, \eta)\) plane.
Relative velocity field  Similar to the symmetric case, governing equation for the relative velocity field in rotating frame is of the same form as given in (3.11). In component form, the equation is given by

\[ \dot{\zeta} = \dot{\theta}\eta - \frac{2\eta}{3\pi[(\zeta + 1/3)^2 + \eta^2]} \left\{ 1 - \exp \left[ \frac{-(\zeta + 1/3)^2 + \eta^2}{4\tau} \right] \right\} \]

\[ - \frac{\eta}{3\pi[(\zeta - 2/3)^2 + \eta^2]} \left\{ 1 - \exp \left[ \frac{-(\zeta - 2/3)^2 + \eta^2}{4\tau} \right] \right\} , \]

\[ \dot{\eta} = -\dot{\theta}\zeta + \frac{2(\zeta + 1/3)}{3\pi[(\zeta + 1/3)^2 + \eta^2]} \left\{ 1 - \exp \left[ \frac{-(\zeta + 1/3)^2 + \eta^2}{4\tau} \right] \right\} \]

\[ + \frac{\zeta - 2/3}{3\pi[(\zeta - 2/3)^2 + \eta^2]} \left\{ 1 - \exp \left[ \frac{-(\zeta - 2/3)^2 + \eta^2}{4\tau} \right] \right\} . \]  

(3.18)

Similar to the absolute velocity field, fixed points must lie on the \( \zeta \) axis, and the non-trivial governing equation for the \( \zeta \) component of fixed points is

\[ -\dot{\theta}\zeta + \frac{2}{3\pi(\zeta + 1/3)} \left\{ 1 - \exp \left[ \frac{-(\zeta + 1/3)^2}{4\tau} \right] \right\} \]

\[ + \frac{1}{3\pi(\zeta - 2/3)} \left\{ 1 - \exp \left[ \frac{-(\zeta - 2/3)^2}{4\tau} \right] \right\} = 0 , \]

where the rotation rate \( \dot{\theta} \) is given in (3.5). The solutions are \((-1/3, 0), (2/3, 0), (\zeta^*_f1, 0), (\zeta^*_f2, 0)\) and \((\zeta^*_3, 0)\). The horizontal components of the latter three fixed points are plotted in Figure 3.27. The fixed point \((\zeta^*_f1, 0)\) starts from \((-1.0816, 0)\) and eventually converges to \((-1/3, 0)\). The fixed point \((\zeta^*_f2, 0)\) starts from \((0.2689, 0)\), then coincides with another fixed point \(\xi_R = (2/3, 0)\) at a bifurcation time \(\tau^*_3 \approx 0.052\), and finally collapses with \((\zeta^*_3, 0)\) (which starts from \((1.146, 0)\)) at another bifurcation time \(\tau^*_5 \approx 0.0525\). At \(\tau^*_5\), they vanish at \((\zeta_{\text{vanish}} \approx 0.711, 0)\). Since this
vortex structure is not symmetric about the $\eta$ axis, no fixed point should be on the $\eta$ axis. The other pair of fixed points can be solved analytically, and they are located at $(1/6, \pm \sqrt{3}/2)$.

The hyperbolic and elliptic characters of these fixed points are obtained by linearizing (3.18) about the fixed points and analyzing the eigenvalue problems associated with the linearized system. The linearized equation about all the fixed points can be written in the same matrix form,

$$
\begin{pmatrix}
\delta \dot{\zeta} \\
\delta \dot{\eta}
\end{pmatrix} = 
\begin{bmatrix}
0 & C_{12} \\
C_{21} & 0
\end{bmatrix}
\begin{pmatrix}
\delta \zeta \\
\delta \eta
\end{pmatrix}.
$$

Figure 3.27: Horizontal components of fixed points in relative velocity field for the asymmetric co-rotating pair in the rotating frame. (a) $(\zeta_{f1}^*, 0)$, (b) $(\zeta_{f2}^*, 0)$ and $(\zeta_{f3}^*, 0)$. $(\zeta_{f1}^*, 0)$ initially starts from $(-1.082, 0)$ and converges to $(-1/3, 0)$ as $\tau \to \infty$. $(\zeta_{f2}^*, 0)$ initially starts from $(0.269, 0)$ and coincides with $(2/3, 0)$ at bifurcation time $\tau_{3}^* \approx 0.052$. $(\zeta_{f3}^*, 0)$ initially starts from $(1.146, 0)$ and collapses with $(\zeta_{f2}^*, 0)$ at the bifurcation time $\tau_{5}^* \approx 0.0525$. After $\tau_{5}^*$, they vanish at $(\zeta_{\text{vanish}}^* \approx 0.711, 0)$. 
Eigenvalues are given by $\lambda_{1,2} = \pm \sqrt{C_{12}C_{21}}$. Around $(-1/3, 0)$,

\[ C_{12} = \dot{\theta} - \frac{1}{4\pi\tau} - \frac{1}{2\pi} \left[ 1 - \exp \left(\frac{-1}{4\tau}\right) \right], \]
\[ C_{21} = -\dot{\theta} + \frac{1}{4\pi\tau} \left[ 1 + \exp \left(\frac{-1}{4\tau}\right) \right] - \frac{1}{2\pi} \left[ 1 - \exp \left(\frac{-1}{4\tau}\right) \right]. \]

Real and imaginary parts of the eigenvalues associated with $(-1/3, 0)$ are shown in Figure 3.28(a). One can see that $(-1/3, 0)$ is always an elliptic point. For $(2/3, 0)$,

Figure 3.28: Real and imaginary parts of the eigenvalues associated with $(-1/3, 0)$ and $(2/3, 0)$ for the asymmetric co-rotating pair. $(-1/3, 0)$ is always an elliptic point, and $(2/3, 0)$ is initially an elliptic point, after $\tau_3^*$ it becomes a hyperbolic point.

$C_{12}$ and $C_{21}$ are given by

\[ C_{12} = \dot{\theta} - \frac{1}{8\pi\tau} - \frac{1}{\pi} \left[ 1 - \exp \left(\frac{-1}{4\tau}\right) \right], \]
\[ C_{21} = -\dot{\theta} + \frac{1}{8\pi\tau} \left[ 1 + 4 \exp \left(\frac{-1}{4\tau}\right) \right] - \frac{1}{\pi} \left[ 1 - \exp \left(\frac{-1}{4\tau}\right) \right]. \]

Real and imaginary parts of these eigenvalues are plotted in Figure 3.28(b). It is initially an elliptic point. After $\tau_3^*$, it becomes a hyperbolic point.
Figure 3.29: Real and imaginary parts of the eigenvalues associated with \((\zeta_{f1}^*, 0), (\zeta_{f2}^*, 0)\) and \((\zeta_{f3}^*, 0)\) for the asymmetric co-rotating pair. \((\zeta_{f1}^*, 0)\) is always a hyperbolic point. \((\zeta_{f2}^*, 0)\) is initially a hyperbolic point, after \(\tau^*_3\) it becomes an elliptic point, and after \(\tau^*_5\) it vanishes. \((\zeta_{f3}^*, 0)\) is a hyperbolic point, after \(\tau^*_5\) it vanishes.

For \((\zeta_{fi}^*, 0), i = 1, 2, 3,\) \(C_{12}\) and \(C_{21}\) are given by the same form,

\[
C_{12} = \hat{\theta} - \frac{1}{\pi(\zeta_{fi}^* + 1/3)^2} \left\{ 1 - \exp \left[ -\frac{(\zeta_{fi}^* + 1/3)^2}{4\tau} \right] \right\} - \frac{1}{2\pi(\zeta_{fi}^* - 2/3)^2} \left\{ 1 - \exp \left[ -\frac{(\zeta_{fi}^* - 2/3)^2}{4\tau} \right] \right\},
\]

\[
C_{21} = -\hat{\theta} + \frac{1}{2\pi\tau} \exp \left[ -\frac{(\zeta_{fi}^* + 1/3)^2}{4\tau} \right] + \frac{1}{4\pi\tau} \exp \left[ -\frac{(\zeta_{fi}^* - 2/3)^2}{4\tau} \right] - \frac{1}{\pi(\zeta_{fi}^* + 1/3)^2} \left\{ 1 - \exp \left[ -\frac{(\zeta_{fi}^* + 1/3)^2}{4\tau} \right] \right\} - \frac{1}{2\pi(\zeta_{fi}^* - 2/3)^2} \left\{ 1 - \exp \left[ -\frac{(\zeta_{fi}^* - 2/3)^2}{4\tau} \right] \right\}.
\]

Real and imaginary parts of eigenvalues associated with the three fixed points are plotted in Figure 3.29. The left fixed point \((\zeta_{f1}^*, 0)\) is always a hyperbolic point. The middle one \((\zeta_{f2}^*, 0)\) is initially a hyperbolic point, after \(\tau^*_3\) it becomes an elliptic
point, and at $\tau^*_5$ it collapses with $(\zeta^*_f, 0)$ and vanishes. The right one $(\zeta^*_f, 0)$ is initially a hyperbolic point, and after $\tau^*_5$ it vanishes.

The remaining pair is $(1/6, \pm \sqrt{3}/2)$, and from the streamline plot in Figure 3.26, one can easily see they are always elliptic points. Just like the symmetric case, all eigenvalues eventually approach 0 due to viscosity.

![Separatrices Evolution](image)

**Figure 3.30:** Evolution of separatrices associated with the relative velocity field of the asymmetric co-rotating pair in rotating frame. The arrows on separatrices show velocity direction. Hyperbolic points are represented by cross-sections of separatrices. Elliptic points are small circles.

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Figure 3.31: (a) Dye visualization of an asymmetric co-rotating vortex pair’s merging process from Trieling et al. [101]. (b) Vorticity contours of an asymmetric merger from the Navier-Stokes simulation in Brandt & Nomura [16].

The bifurcation sequence of the asymmetric case is plotted in Figure 3.30. Initially, 7 stagnation points exist: 3 hyperbolic points and 4 elliptic points, see Figure 3.30(a). As time evolves, the 1st, 2nd and 4th bifurcations are due to the collapsing of separatrices; while the 3rd and 5th bifurcations are due to the colliding of instantaneous stagnation points; these 5 bifurcations occur within a very short period of time ($\tau^*_1 \approx 0.0510, \tau^*_2 \approx 0.0511, \tau^*_3 \approx 0.0520, \tau^*_4 \approx 0.0524$ and $\tau^*_5 \approx 0.0525$). The topology of separatrices remains the same as Figure 3.30(k) after the 5th bifurcation time. The last bifurcation occurs only when $\tau \to \infty$, though the physical meaning of the instantaneous stagnation points and separatrices diminishes now since the velocity field is zero everywhere.

**Passive particle evolution** The experimental result of an asymmetric merger from Trieling et al. [101] is shown again in Figure 3.31(a). And vorticity contours of
Figure 3.32: Passive particle evolution for the asymmetric case.

Similar to the symmetric case, we let the vortex cores diffuse for a short amount of time \( \tau_0 = 0.002 \), hence the core is \( a_0 = \sqrt{4 \tau_0} \approx 0.089 \), then seed the flow with passive particles of six different colors as shown in Figure 3.32(a). As time evolves, the inner separatrix shrinks and the green particles leak into the exchange band and circulate around the inner cores, see Figure 3.32(b). As the inner separatrix continue to shrink, all of the green particles will enter the exchange band, followed by all the teal particles and some of the blue particles, while the left particles still remain inside the inner core and rotate around the left elliptic point. This is because in this case, the right ring of the inner separatrix shrinks, but the left ring actually expands. As the middle and right hyperbolic points continue to approach
the right elliptic point, some of the particles leak along the unstable manifold. As mentioned before, the 5 bifurcation times occur within a short period of time, hence the positions of the passive particles will not change much during the bifurcations. As shown in Figure 3.32(c), the exchange band disappears, all of the left particles remain in the inner core, while some of the right particles circulate and the others leak to the outer recirculation area or the outer area. The majority of blue particles remain close to the now hyperbolic point. In Figure 3.32(d) and (e), the blue particles escape along the unstable manifolds, and the right particles erode into the band between the outer and inner separatrices and form filaments. As the left hyperbolic point continues to approach the left elliptic point, the left ring of the inner separatrix continues to shrink, and the purple particles start to leak outside the left ring, see Figure 3.32(f). Since the left ring will shrink to zero only when $\tau \to \infty$, it will take infinite amount of time for the red particles to completely leak along the unstable manifold associated with the left hyperbolic point.

**Summary**  The homotopic equivalences of the bifurcations of the separatrices in the absolute velocity field and relative velocity field are plotted in Figure 3.33 and 3.34, respectively. Bifurcation states are depicted in boxes. The timeline of all

![Figure 3.33](image)

Figure 3.33: Homotopic equivalences of the separatrices in the absolute velocity field for the asymmetric co-rotating pair.
Figure 3.34: Homotopic equivalences of the separatrices in the relative velocity field for the asymmetric co-rotating pair.

important events for the asymmetric co-rotating pair based on the multi-Gaussian model is plotted in Figure 3.35.

Figure 3.35: Timeline of important events for the asymmetric co-rotating vortex pair.
3.3 Vortex tripole

Figure 3.36: Schematics of a vortex tripole.

In inviscid flow, a vortex tripole consists of three equally spaced collinear point vortices, two vortices of the same strength $-\Gamma$ are on the sides and one of strength $2\Gamma$ is at the center. The net circulation is zero. It is a relative equilibrium. Initially, $z_L(0) = -1 + i0$, $z_C(0) = 0 + i0$ and $z_R(0) = 1 + i0$. The tripole rotates around the center vortex at a constant rotation rate

$$\dot{\theta} = \frac{3}{4\pi}.$$  

Dynamics of vortex centers Now we consider the problem in viscous fluid using the multi-Gaussian model. The vorticity field is given by

$$\omega = \frac{1}{4\pi \tau} \left[ -\exp\left(\frac{-\|z - z_L\|^2}{4\tau}\right) + 2\exp\left(\frac{-\|z - z_C\|^2}{4\tau}\right) - \exp\left(\frac{-\|z - z_R\|^2}{4\tau}\right) \right].$$
Similar to the previous cases, the structure of vortex centers does not change its shape, but the system rotates around the center vortex. Due to symmetry, $\dot{z}_C \equiv 0$.

The rotation rate $\dot{\theta}$ can be obtained by focusing on $z_R$:

$$
\dot{z}_R = -i \dot{\theta} e^{-i \theta} = -\frac{1}{4\pi i e^{i \theta}} \left[ 1 - \exp \left( \frac{-1}{\tau} \right) \right] + \frac{1}{\pi i e^{i \theta}} \left[ 1 - \exp \left( \frac{-1}{4\tau} \right) \right].
$$

Therefore, the rotation rate is given by

$$
\dot{\theta} = \frac{1}{4\pi} \left[ 3 + \exp \left( \frac{-1}{\tau} \right) - 4 \exp \left( \frac{-1}{4\tau} \right) \right]. \quad (3.19)
$$

Integrating $\dot{\theta}$ in time, one has the orientation angle $\theta$,

$$
\theta = \frac{Re}{4\pi} \left[ 3\tau + \exp \left( \frac{-1}{\tau} \right) \tau - 4 \exp \left( \frac{-1}{4\tau} \right) \tau + \text{Ei} \left( \frac{-1}{\tau} \right) - \text{Ei} \left( \frac{-1}{4\tau} \right) \right]. \quad (3.20)
$$
Figure 3.37 shows $\dot{\theta}$ and $\theta$ as functions of time $\tau$. Just like the vortex pair cases, initially, $\dot{\theta}(0)$ is the same value as the inviscid case, then decays gradually to zero as time goes to infinity.

**Vorticity and absolute velocity field**  Vorticity along the connecting line of the vortex centers is plotted in Figure 3.38. We express the absolute velocity field in a frame $\xi = (\zeta, \eta)$ co-rotating with the vortex tripole at the time dependent rotation rate $\dot{\theta}$ given by (3.19). The positions of the vortex centers in this frame remain $\xi_L = (-1, 0)$, $\xi_C = (0, 0)$ and $\xi_R = (1, 0)$ for all time. Velocity $\mathbf{v} = (v_\zeta, v_\eta)$
with respect to the inertial frame but expressed in the rotating frame is given in component form as follows

\[
v_\zeta = - \frac{\eta}{\pi (\zeta^2 + \eta^2)} \left[ 1 - \exp \left( -\frac{\zeta^2 + \eta^2}{4\tau} \right) \right] + \frac{\eta}{2\pi [((\zeta + 1)^2 + \eta^2)]} \left\{ 1 - \exp \left( -\frac{(\zeta + 1)^2 + \eta^2}{4\tau} \right) \right\} + \frac{\eta}{2\pi [((\zeta - 1)^2 + \eta^2)]} \left\{ 1 - \exp \left( -\frac{(\zeta - 1)^2 + \eta^2}{4\tau} \right) \right\},
\]

(3.21)

\[
v_\eta = \frac{\zeta}{\pi (\zeta^2 + \eta^2)} \left[ 1 - \exp \left( -\frac{\zeta^2 + \eta^2}{4\tau} \right) \right] - \frac{\zeta + 1}{2\pi [((\zeta + 1)^2 + \eta^2)]} \left\{ 1 - \exp \left( -\frac{(\zeta + 1)^2 + \eta^2}{4\tau} \right) \right\} - \frac{\zeta - 1}{2\pi [((\zeta - 1)^2 + \eta^2)]} \left\{ 1 - \exp \left( -\frac{(\zeta - 1)^2 + \eta^2}{4\tau} \right) \right\}.
\]

The streamlines associated with (3.21) is plotted in Figure 3.39. The flow field does not eventually evolve into a single Gaussian vortex, recall that Gallay & Wayne’s conclusion only applies on non zero vorticity field. Clearly from Figure 3.39, fixed

![Streamlines plots](image)

(a) \(\tau = 0\)  (b) \(\tau = 0.1\)  (c) \(\tau = 0.2\)  (d) \(\tau = 0.3\)

**Figure 3.39:** Streamlines in rotating frame at various instants \(\tau = 0, 0.1, 0.2\) and 0.3. Each plot shows a \([-3, 3] \times [-3, 3]\) window in the \((\zeta, \eta)\) plane.
points lie on the $\zeta$ axes. The governing equation of $\zeta$ in fixed points is given by

$$
\frac{2}{\zeta} \left[ 1 - \exp \left( -\frac{\zeta^2}{4\tau} \right) \right] - \frac{1}{\zeta + 1} \left[ 1 - \exp \left( -\frac{(\zeta + 1)^2}{4\tau} \right) \right] - \frac{1}{\zeta - 1} \left[ 1 - \exp \left( -\frac{(\zeta - 1)^2}{4\tau} \right) \right] = 0.
$$

One finds a total of three fixed points: $(0, 0)$ and $(\pm \zeta_f, 0)$. The horizontal components $\pm \zeta_f$ are plotted in Figure 3.40. One can see that $(\pm \zeta_f, 0)$ start from $(\pm 1, 0)$, and eventually go to $(\pm \infty, 0)$ as $\tau \to \infty$. Obviously, the three fixed points are always elliptic points, and there is no separatrix in the absolute velocity field.
**Relative velocity field**  Similar to the previous cases, governing equations for the relative velocity field in rotating frame are given by

\[
\begin{align*}
\dot{\zeta} &= \dot{\theta} \eta - \frac{\eta}{\pi(\zeta^2 + \eta^2)} \left[ 1 - \exp \left( -\frac{\zeta^2 + \eta^2}{4\tau} \right) \right] \\
&\quad + \frac{\eta}{2\pi[(\zeta + 1)^2 + \eta^2]} \left\{ 1 - \exp \left[ -\frac{(\zeta + 1)^2 + \eta^2}{4\tau} \right] \right\} \\
&\quad + \frac{\eta}{2\pi[(\zeta - 1)^2 + \eta^2]} \left\{ 1 - \exp \left[ -\frac{(\zeta - 1)^2 + \eta^2}{4\tau} \right] \right\}, \\
\end{align*}
\]

\[
\begin{align*}
\dot{\eta} &= -\dot{\theta} \zeta + \frac{\zeta}{\pi(\zeta^2 + \eta^2)} \left[ 1 - \exp \left( -\frac{\zeta^2 + \eta^2}{4\tau} \right) \right] \\
&\quad - \frac{\zeta + 1}{2\pi[(\zeta + 1)^2 + \eta^2]} \left\{ 1 - \exp \left[ -\frac{(\zeta + 1)^2 + \eta^2}{4\tau} \right] \right\} \\
&\quad - \frac{\zeta - 1}{2\pi[(\zeta - 1)^2 + \eta^2]} \left\{ 1 - \exp \left[ -\frac{(\zeta - 1)^2 + \eta^2}{4\tau} \right] \right\}.
\end{align*}
\] (3.22)

Due to symmetry, fixed points must lie on \( \zeta \) and/or \( \eta \) axis. On the \( \zeta \) axis, governing

\[ 0 = \frac{\zeta}{\pi(\zeta^2 + \eta^2)} \left[ 1 - \exp \left( -\frac{\zeta^2 + \eta^2}{4\tau} \right) \right] \]

\[ - \frac{\zeta + 1}{2\pi[(\zeta + 1)^2 + \eta^2]} \left\{ 1 - \exp \left[ -\frac{(\zeta + 1)^2 + \eta^2}{4\tau} \right] \right\} \]

\[ - \frac{\zeta - 1}{2\pi[(\zeta - 1)^2 + \eta^2]} \left\{ 1 - \exp \left[ -\frac{(\zeta - 1)^2 + \eta^2}{4\tau} \right] \right\}.
\]

Figure 3.41: Streamlines associated with relative velocity field for the vortex tripole in rotating frame at various instants \( \tau = 0, 0.15, 0.3 \) and 1. Each plot shows a \([-3, 3] \times [-3, 3]\) window in \((\zeta, \eta)\) plane.
equation for the $\zeta$ component of the fixed points is

$$-\dot{\zeta} + \frac{2}{\pi \zeta} \left[ 1 - \exp \left( -\frac{\zeta^2}{4\tau} \right) \right] - \frac{1}{2\pi (\zeta + 1)} \left\{ 1 - \exp \left[ -\frac{(\zeta + 1)^2}{4\tau} \right] \right\} - \frac{1}{2\pi (\zeta - 1)} \left\{ 1 - \exp \left[ -\frac{(\zeta - 1)^2}{4\tau} \right] \right\} = 0.$$  

The solutions of fixed points on $\zeta$ axis are $(0, 0)$ and $(\pm 1, 0)$, which correspond to the vortex centers. On the $\eta$ axis, governing equation for $\eta$ component is

$$\dot{\eta} + \frac{\eta}{\pi (1 + \eta^2)} \left[ 1 - \exp \left( -\frac{1 + \eta^2}{4\tau} \right) \right] - \frac{1}{\pi \eta} \left[ 1 - \exp \left( -\frac{\eta^2}{4\tau} \right) \right] = 0.$$  

The solutions are $(0, 0)$ and $(0, \pm \eta^*_f)$. Figure 3.42 shows the $\eta$ components of the pair $(0, \pm \eta^*_f)$ as functions of time. One can see that $\eta^*_f$ start from $\sqrt{(\sqrt{57} - 3)/6} \approx 0.87$.

![Figure 3.42](image_url)

Figure 3.42: Vertical components of the fixed points $(0, \pm \eta^*_f)$. They collapse at the origin at $\tau_b^* \approx 0.236$.

The pair eventually collapses at the origin at a finite bifurcation time $\tau_b^* \approx 0.236$. Before $\tau_b$, the system has 5 fixed points: $(0, 0)$, $(\pm 1, 0)$ and $(0, \pm \eta^*_f)$; after $\tau_b$, the system has 3 fixed points: $(0, 0)$ and $(\pm 1, 0)$.
Similar to the previous cases, the linearized equation associated with the fixed points can be written in a common matrix form,

\[
\begin{pmatrix}
\delta \dot{\zeta} \\
\delta \dot{\eta}
\end{pmatrix} = \begin{bmatrix} 0 & C_{12} \\ C_{21} & 0 \end{bmatrix} \begin{pmatrix} \delta \zeta \\ \delta \eta \end{pmatrix},
\]

and the eigenvalues are given by \( \lambda_{1,2} = \pm \sqrt{C_{12}C_{21}} \). For \((0,0)\),

\[
C_{12} = \dot{\theta} - \frac{1}{4\pi \tau} + \frac{1}{\pi} \left[ 1 - \exp \left( -\frac{1}{4\tau} \right) \right],
\]

\[
C_{21} = -\dot{\theta} + \frac{1}{4\pi \tau} - \frac{1}{2\pi \tau} \exp \left( -\frac{1}{4\tau} \right) + \frac{1}{\pi} \left[ 1 - \exp \left( -\frac{1}{4\tau} \right) \right].
\]

Real and imaginary parts of the eigenvalues associated with the origin are shown in Figure 3.43. Initially the origin is an elliptic point, after the bifurcation time \( \tau^*_b \),

![Figure 3.43](image)

Figure 3.43: Real and imaginary parts of the eigenvalues of \((0,0)\) for the vortex tripole. Before \( \tau^*_b \), the origin is an elliptic point, after \( \tau^*_b \), it becomes a hyperbolic point.
it becomes a hyperbolic point. For $(\pm 1, 0)$, $C_{12}$ and $C_{21}$ are now given by

$$C_{12} = \dot{\theta} + \frac{1}{8\pi \tau} - \frac{1}{8\pi} \left[ 7 + \exp \left( \frac{-1}{\tau} \right) - 8 \exp \left( \frac{-1}{4\tau} \right) \right],$$

$$C_{21} = -\dot{\theta} - \frac{1}{8\pi} \left[ 7 + \exp \left( \frac{-1}{\tau} \right) - 8 \exp \left( \frac{-1}{4\tau} \right) \right] - \frac{1}{8\pi \tau} \left[ 1 + 2 \exp \left( \frac{-1}{\tau} \right) - 4 \exp \left( \frac{-1}{4\tau} \right) \right].$$

Real and imaginary parts of these eigenvalues are plotted in Figure 3.44. These two fixed points are always elliptic points. For $(0, \pm \eta_f^*)$, $C_{12}$ and $C_{21}$ are given by

$$C_{12} = \dot{\theta} + \frac{1}{\pi \eta_f^2} \left[ 1 - \left( \frac{1 + \eta_f^2}{2\tau} \right) \exp \left( \frac{-\eta_f^2}{4\tau} \right) \right]
+ \frac{1}{\pi (1 + \eta_f^2)} \left\{ \left[ 1 - \frac{\eta_f^2}{1 + \eta_f^2} \right] + \left( \frac{\eta_f^2}{2\tau} - \frac{1 - \eta_f^2}{1 + \eta_f^2} \right) \exp \left( -\frac{1 + \eta_f^2}{4\tau} \right) \right\},$$

$$C_{21} = -\dot{\theta} + \frac{1}{\pi \eta_f^2} \left[ 1 - \exp \left( \frac{-\eta_f^2}{4\tau} \right) \right]
+ \frac{1}{\pi (1 + \eta_f^2)} \left\{ \left[ 1 - \frac{\eta_f^2}{1 + \eta_f^2} \right] - \left( \frac{1}{2\tau} + \frac{1 - \eta_f^2}{1 + \eta_f^2} \right) \exp \left( -\frac{1 + \eta_f^2}{4\tau} \right) \right\}. $$

Figure 3.44: Real and imaginary parts of the eigenvalues of $(\pm 1, 0)$ for the vortex tripole. They are always elliptic points.
Real and imaginary parts of the eigenvalues are shown in Figure 3.45 as functions of time. This pair remains hyperbolic points for all time. Indeed, after \( \tau_b^* \), \((0, \pm \eta_f)\) collapse at the origin, the part after \( \tau_b^* \) in Figure 3.45 is identical to the part after \( \tau_b^* \) in Figure 3.43, which corresponds to a hyperbolic point.

We have shown the fixed points and the associated eigenvalues for the vortex tripole in its rotating frame. Notice that all eigenvalues eventually become zero because the velocity field eventually diffuses to zero due to viscosity. One can draw instantaneous streamlines based on the velocity field given by (3.22). The evolution of the fixed points and separatrices at four instants \( \tau = 0, 0.06, \tau_b^* \approx 0.236, \) and 0.5 are illustrated in Figure 3.46. At \( \tau = 0 \), the separatrix divides the fluid field into four regions: three regions around the three vortices, and the fourth region is the rest of fluid outside the separatrix, see Figure 3.46(a). When \( 0 < \tau < \tau_b^* \), the hyperbolic points approach the origin, and the separatrix is shrinking. As an example, the separatrix is shown at time \( \tau = 0.06 \) in Figure 3.46(b). At the bifurcation time \( \tau_b^* \approx 0.236 \) in Figure 3.46(c), the hyperbolic points collapse at the origin, and the

![Figure 3.45: Real and imaginary parts of the eigenvalues of \((0, \pm \eta_f)\) for the vortex tripole.](image-url)
region around the origin shrinks to zero, only three regions are left in the flow field. At this time, the slope of the separatrix at the origin is perpendicular to the \( \zeta \) axis. Figure 3.46(d) shows the separatrix at \( \tau = 0.5 \) as an example of \( \tau > \tau_b^* \). As \( \tau \to \infty \), the slope of separatrix at the origin eventually approaches zero, and the separatrix eventually shrinks to zero.

Figure 3.46: Separatrices in relative velocity field for vortex tripole in rotating frame. The arrows on separatrices show velocity directions. Hyperbolic points are represented by cross-sections of separatrices. Elliptic points are small circles.

Topologically, \( \tau_b^* \) defines the bifurcation time scale of the topological change of the streamlines. This change is summarized by the homotopic equivalence in Figure 3.47. The bifurcation state is depicted in box.

Figure 3.47: Homotopic equivalences of the separatrices for the vortex tripole.
3.4 Collinear three vortex fixed equilibrium

In inviscid flow, a collinear three vortex fixed equilibrium ("fixed equilibrium" for short) consists of three equally spaced collinear point vortices, two vortices of the same strength $2\Gamma$ at the sides and one vortex of strength $-\Gamma$ at the center. It is a fixed equilibrium. Unlike the vortex tripole, the net circulation in this case is *not* zero. In dimensionless form, the vortex centers are located at $z_L = -1 + i0$, $z_C = 0 + i0$ and $z_R = 1 + i0$ for all time in inviscid flow. In another word, $\dot{\theta} = 0$ for all time.

**Dynamics of vortex centers**  Now we consider the problem in viscous fluid using the multi-Gaussian model. The vorticity field is given by

$$\omega = \frac{1}{4\pi \tau} \left[ 2 \exp \left( -\frac{\|z - z_L\|^2}{4\tau} \right) - \exp \left( -\frac{\|z - z_C\|^2}{4\tau} \right) + 2 \exp \left( -\frac{\|z - z_R\|^2}{4\tau} \right) \right].$$

Similar as before, the induced velocities of the vortex centers are always perpendicular to their connecting line, and the system rotates around the origin. Due to
symmetry, \( \dot{z}_C = 0 \) for all time. The rotation rate \( \dot{\theta} \) can be obtained by focusing on \( z_R \):

\[
\dot{z}_R^* = -\dot{\theta} e^{-i\theta} = \frac{2}{4\pi i e^{i\theta}} \left[ 1 - \exp \left( -\frac{1}{\tau} \right) \right] - \frac{1}{2\pi i e^{i\theta}} \left[ 1 - \exp \left( -\frac{1}{4\tau} \right) \right].
\]

Therefore, the rotation rate is given by

\[
\dot{\theta} = \frac{1}{2\pi} \left[ \exp \left( -\frac{1}{4\tau} \right) - \exp \left( -\frac{1}{\tau} \right) \right]. \tag{3.23}
\]

Integrating \( \dot{\theta} \) in time, one has the orientation angle \( \theta \),

\[
\theta = \frac{Re}{2\pi} \left[ \exp \left( -\frac{1}{4\tau} \right) \tau - \exp \left( -\frac{1}{\tau} \right) \tau + \frac{1}{4} Ei \left( -\frac{1}{4\tau} \right) - Ei \left( -\frac{1}{\tau} \right) \right]. \tag{3.24}
\]

Figure 3.49 shows \( \dot{\theta} \) and \( \theta \) as functions of time \( \tau \). The rotation rate starts from zero, and eventually decays to zero again as \( \tau \to \infty \). At the intermediate time \( \tau_{\text{max}} \), the
rotation rate reaches the maximum $\dot{\theta}_{\text{max}}$, which can be calculated by setting the derivative of $\dot{\theta}$ to be zero,

$$\frac{d\dot{\theta}}{d\tau} = \frac{1}{2\pi} \left[ \frac{1}{4\tau^2} \exp\left(-\frac{1}{4\tau}\right) - \frac{1}{\tau^2} \exp\left(-\frac{1}{\tau}\right) \right] = 0,$$

which gives

$$\tau_{\text{max}} = \frac{3}{8 \ln 2} \approx 0.541, \quad \dot{\theta}_{\text{max}} = \frac{1}{2\pi} \left[ \exp\left(-\frac{2\ln 2}{3}\right) - \exp\left(-\frac{8\ln 2}{3}\right) \right] \approx 0.075.$$  

Hence, we can view that $\tau_{\text{max}}$ is a measurement of the spatial scale of this structure $(b_0)$, and $\dot{\theta}_{\text{max}}$ is a measurement of both the net circulation and spatial distribution of the structure ($\Gamma$ and $b_0$). Finally, the positions of the vortex centers are given by

$$z_L = -\cos \theta - i \sin \theta, \quad z_C = 0 + i 0, \quad z_R = \cos \theta + i \sin \theta.$$  

(3.25)

**Vorticity and absolute velocity field**  The vorticity contours and streamlines of the fixed equilibrium in inertial frame is depicted in Figure 3.50 at various instants. Similar to the co-rotating pairs, the flow field eventually evolves into a single Gaussian vortex. While the structure of the system is evolving, it is also rotating unsteadily, and the rotation rate is given by (3.23). We express the absolute velocity field in a frame $\xi = (\zeta, \eta)$ co-rotating with the fixed equilibrium at the time-dependent rotation rate $\dot{\theta}$. The locations of vortex centers in this frame remain $\xi_L = (-1, 0)$, $\xi_C = (0, 0)$ and $\xi_R = (1, 0)$ for all time. The absolute velocity
\( \mathbf{v} = (v_\zeta, v_\eta) \) with respect to the inertial frame but expressed in the rotating frame can be given in component form,

\[
\begin{align*}
    v_\zeta &= \frac{\eta}{2\pi(\zeta^2 + \eta^2)} \left[ 1 - \exp \left( -\frac{\zeta^2 + \eta^2}{4\tau} \right) \right] \\
    &\quad - \frac{\eta}{\pi[(\zeta + 1)^2 + \eta^2]} \left\{ 1 - \exp \left[ -\frac{(\zeta + 1)^2 + \eta^2}{4\tau} \right] \right\} \\
    &\quad - \frac{\eta}{\pi[(\zeta - 1)^2 + \eta^2]} \left\{ 1 - \exp \left[ -\frac{(\zeta - 1)^2 + \eta^2}{4\tau} \right] \right\}, \\
    v_\eta &= -\frac{\zeta}{2\pi(\zeta^2 + \eta^2)} \left[ 1 - \exp \left( -\frac{\zeta^2 + \eta^2}{4\tau} \right) \right] \\
    &\quad + \frac{\zeta + 1}{\pi[(\zeta + 1)^2 + \eta^2]} \left\{ 1 - \exp \left[ -\frac{(\zeta + 1)^2 + \eta^2}{4\tau} \right] \right\} \\
    &\quad + \frac{\zeta - 1}{\pi[(\zeta - 1)^2 + \eta^2]} \left\{ 1 - \exp \left[ -\frac{(\zeta - 1)^2 + \eta^2}{4\tau} \right] \right\}.
\end{align*}
\]

The streamlines associated with (3.26) are plotted in Figure 3.51. The separatrices
Figure 3.51: Streamlines in rotating frame at various instants $\tau = 0, \tau_1 \approx 0.064, 0.2$ and $\tau_2 \approx 0.344$. Each plot shows a $[-2.5, 2.5] \times [-2.5, 2.5]$ window in the $(\zeta, \eta)$ plane. Instantaneous separatrices are depicted in thick lines with arrows representing directions of flow.

are depicted in thick lines with arrows representing the direction of the flow. The fixed points of the velocity field are obtained by setting the right-hand-sides of (3.26) to zero. Due to symmetry of the velocity field, the fixed points must lie on the $\zeta$ and $\eta$ axes. One finds a total of 5 fixed points: one initial elliptic point at the origin, a pair of initial hyperbolic points at $(0, \pm \eta_f)$ and a pair of initial elliptic points at $(\pm \zeta_f, 0)$, where $\pm \eta_f$ and $\pm \zeta_f$ need to be obtained numerically, see Figure 3.52.

At time $\tau = 0$, the streamlines are identical to that of the inviscid equilibrium, with a separatrix linking the two hyperbolic stagnations points on the $\eta$ axis, as

Figure 3.52: Fixed points: (a) vertical components of $(0, \pm \eta_f)$, they collide at the origin at bifurcation time $\tau_1 \approx 0.064$; (b) horizontal components of $(\pm \zeta_f, 0)$, they collide at the origin at bifurcation time $\tau_2 \approx 0.344$. 

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shown in Figure 3.51(a). Initially, the separatrix divides the fluid domain into four regions: three regions, one around each vortex, and a fourth region bounded by the separatrix and the bound at infinity. As time evolves, the location of the fixed points change as shown in Figure 3.52, and the separatrix evolves accordingly. The pair of initial hyperbolic points \((0, \pm \eta_f)\) start from \((0, \pm \sqrt{3}/3)\) then collide together with the elliptic point at the origin in finite time \(\tau_1 \approx 0.064\) to transform the origin into a hyperbolic point. This collision is accompanied by a change in the streamline topology where the region around the center vortex disappears, see Figure 3.51(b). Time \(\tau_1\) is referred to as the first bifurcation time. Similarly, \((\pm \zeta_f, 0)\) starts from \((\pm 1, 0)\) and collides at the now hyperbolic point at the origin at time \(\tau_2 \approx 0.344\). Time \(\tau_2\) is referred to as the second bifurcation time. For \(\tau > \tau_2\), one has one single elliptic point at the origin as expected from the asymptotic Lamb-Oseen solution.

**Comparison to Navier-Stokes solution**

Up to now, we have only analyzed the viscous system using our multi-Gaussian model. Naturally, one would like to validate this model by comparing the result with the exact solution of Navier-Stokes equation. We present a direct numerical simulation of the system of vorticity equations (2.6) and velocity equations (2.7) subject to the initial condition

\[
\omega(z, 0) = 2\Gamma \delta [z - (-1 + i0)] - \Gamma \delta(z) + 2\Gamma \delta [z - (1 + i0)] .
\]
We use the numerical algorithm devised in [21] that utilizes a second-order finite difference method with a multi-domain non-reflecting boundary condition to emulate the infinite fluid domain. This is a mesh-based method which poses a problem in handling the Dirac delta initial conditions because they are not well posed for discretization on a standard Euclidean mesh. To overcome this problem, we consider the initial conditions of the vorticity field as a superposition of three slightly diffused Gaussian peaks. In all the simulations presented here, we diffuse the initial Dirac delta vorticity field by $\epsilon$ such that $\epsilon/Re = 2 \times 10^{-4}$, e.g., $\epsilon = 0.2$ when $Re = 1000$. This, of course, introduces a slight mismatch in the initial conditions used for the numerical simulation with those used in the model, an error we cannot completely eliminate, but should be kept in mind when comparing the simulation with the model. We compute the time evolution of the vorticity field in the window $[-3, 3] \times [-3, 3]$ while the non-reflecting boundary conditions are imposed using the multi-domain technique with 10 nested domains, the largest of which is $2^{10}$ times the size of our result window. The spatial and time steps are set to $\Delta x = \Delta y = 0.01$, $\Delta t = 0.02$.

Some words of caution are in order here, as we will be comparing numerical simulations of the Navier-Stokes equations with our model, and to do so requires that one is able to compare the direct numerical simulation (DNS) Reynolds number with the “model” Reynolds number. For this, it is better to think of the Reynolds number as the ratio of inertial effects $-\mathbf{u} \cdot \nabla \omega$ over diffusive effects $\Delta \omega$. In some sense, one can think of the term $-\mathbf{u} \cdot \nabla \omega$ as being primarily responsible for the
rotation we will discuss, while the term $\Delta \omega$ not only triggers the rotation, but diffuses the cores of the vortices. For any DNS, this creates an ‘effective’ Reynolds number which depends not only on the choice $\Gamma/\nu$, but also on numerical discretization since it affects the “rotation” and “diffusion”. The “model” Reynolds number will be discussed in more detail in the upcoming analysis. Figure 3.53 depicts the

![Vorticity contours (top row) and streamlines (bottom row) of Navier-Stokes simulation for $Re = 1000$ at $t = 0, 2.8, 43.7$ and $47.4$. The vortex configuration rotates unsteadily for $t > 0$. The center vortex stretches and diffuses out first, then the outer two vortices begin to merge. Eventually the vortex configuration approach a single Gaussian vortex.](image)

Figure 3.53: Vorticity contours (top row) and streamlines (bottom row) of Navier-Stokes simulation for $Re = 1000$ at $t = 0, 2.8, 43.7$ and $47.4$. The vortex configuration rotates unsteadily for $t > 0$. The center vortex stretches and diffuses out first, then the outer two vortices begin to merge. Eventually the vortex configuration approach a single Gaussian vortex.

Comparing the Navier-Stokes solution and the result from multi-Gaussian model shown in Figure 3.53, one notices the following common characters: (i) an unsteady rotation of the whole vortex configuration for time greater than zero, (ii) a stretching of the middle vortex, and (iii) eventual merging of the outer two vortices to form one single-peaked Gaussian vortex. However, here some care is in order, as clearly
Figures 3.50(b)-(d) (model) and Figures 3.53(b)-(d) (DNS) show some important differences. Not only are the timescales different, but Figure 3.53(b) shows a convective “wrapping” and “stretching” of the middle vortex around the outer two before the diffusive effects kick in, whereas Figure 3.50(b) shows the stretching, but not the wrapping. Here it is important to remember that the passively advected field, as shown in Figure 3.66, is an important part of the model, and this field does show some of the same nonlinear wrapping features that appear in the DNS Figure 3.53(b). One could say, in some respects, that the outer two vortices, being twice the strength of the inner one, are the primary “drivers” of the flow field, which is perhaps why Figure 3.53(e)-(h) match relatively well with Figure 3.50(e)-(h). The “passively advected” inner vortex shown in Figure 3.53(b) is better reflected in the passive particle field shown in Figure 3.66 and discussed at length in the following paragraphs. In turn, because the passively advected field in our model is not affecting the vorticity evolution, whereas in the DNS it is, this helps explain why the timescales associated with the two are different. The model is not an exact solution of the Navier-Stokes equations, and this appears to be the main physical manifestation of this fact. The residual $\sigma$ of the model is computed by substituting the solution of (2.27) and (2.28) into the Navier-Stokes equation (2.6). If the solution of the model is also an exact solution of the Navier-Stokes equation for a given set of initial conditions, the residual $\sigma$ is identically 0. In general, $\sigma$ is not zero (see discussions of this in [33]) and it can be viewed as an indication of the inaccuracy of the multi-Gaussian model. The $L_2$ norm of residual is plotted as a function of time
Figure 3.54: (a) $L_2$ norm of residual $\sigma$ (b) $L_2$ norm of the difference in the vorticity field of the multi-Gaussian model and the single peaked Lamb-Oseen vortex with circulation $3\Gamma$ (c) shows the difference in velocity field. Clearly, for long time, the model approaches the single peaked Gaussian but in short time, the multi-Gaussian, while not numerically accurate in comparison to the Navier-Stokes model as indicated in (a), its dynamics is richer than the single Gaussian as indicated in (b) and (c).

in Figure 3.54(a) for the fixed equilibrium. Figure 3.54(a) shows that as $\tau$ increases, the $L_2$ norm of $\sigma$ tends to zero, indicating that the multi-Gaussian model agrees with the Navier-Stokes solution for $\tau$ large. From the result of Gallay & Wayne and since the total circulation of the initial vorticity field is 3, we know as $\tau \to \infty$ the Navier-Stokes solution approaches a single Gaussian vorticity distribution

$$\omega_\infty = \frac{3}{4\pi\tau} \exp\left(-\frac{\|z\|^2}{4\tau}\right).$$

We compute the difference between the multi-Gaussian model and this asymptotic solution $\omega_\infty$. Figure 3.54(b) and (c) show the $L_2$ norm of the difference in both vorticity and velocity, respectively. These plots confirm that the multi-Gaussian model approaches the asymptotic Lamb-Oseen solution for large time but at the intermediate times, the multi-Gaussian model exhibits richer dynamics than the asymptotic Lamb-Oseen solution. While the dynamics of the multi-Gaussian model
at these intermediate time scales does not faithfully track the Navier-Stokes solution (as seen from Figure 3.54(a)), it does capture more details at the intermediate time scales than the asymptotic Lamb-Oseen vortex and its evolution seems to exhibit the main qualitative features of the Navier-Stokes model as argued next. It is evident

![Graphs showing comparison between Navier-Stokes simulations and multi-Gaussian model]

**Figure 3.55:** Comparison of rotation angle $\theta$ between (a) the Navier-Stokes simulations and (b) the multi-Gaussian model. Navier-Stokes simulations are conducted with the same initial vorticity field for Reynolds numbers $Re = 1000, 2000, 3000$ and $4000$. The results of the multi-Gaussian model are obtained for $\Gamma = 1$ and $\nu = 1/100, 1/200, 1/300, 1/400$. The trend of both models is qualitatively similar.

from Figures 3.53 and 3.50 that both the Navier-Stokes equations and the multi-Gaussian model exhibit a viscosity-induced rotation as time evolves. In Figure 3.55 we compare the qualitative trends of rotation angle $\theta$ obtained from the numerical solution to the Navier-Stokes equation and the analytical solution of multi-Gaussian model. In the Navier-Stokes solution, the rotation angle $\theta$ is obtained by computing the angle between the line traced by the vorticity peaks (see Figure 3.53) and the $x$ axis while it is given by (3.24) in the multi-Gaussian model. Clearly, both the Navier-Stokes solution and the model, although quantitatively distinct, exhibit similar qualitative trends in that the rotation angle $\theta$ is smaller when $Re$ increases.
(in Navier-Stokes) or equivalently when $\Gamma/\nu$ increases (in the model, recall that $\tau = \nu t$). As cautioned earlier about comparing DNS Reynolds numbers with the “model” Reynolds number, if both are thought of strictly as $\Gamma/\nu$, we can only claim qualitative overlap with the model and DNS. To obtain more quantitative overlap would require more effort on our part to obtain an accurate Lagrangian based DNS to get a more detailed handle on the effective numerical Reynolds number, along with a modified model system that does more to couple rotational effects with diffusive effects, neither of which are the immediate goals of the current work. To

quantify the difference between the Navier-Stokes solution and the multi-Gaussian model, we focus on comparing the first bifurcation time $\tau_1$ in the Navier-Stokes simulation for different $Re$ to the first bifurcation time in the model. The result is plotted in Log-Log scale in Figure 3.56 for $Re = 100, 500, 1000, 2000, 3000, 4000, 5000$ and $5000$ (plotted in squares). The dashed line is the best fit straight line using the least square distance rule. The fitted line can be expressed as $\ln(\tau_1) = \ln(Re) - 3.97$, 

Figure 3.56: Comparison of the times of the first bifurcation given by the Navier-Stokes simulations and the multi-Gaussian model in Log-Log plot. Simulation results are plotted as squares for $Re = 100, 500, 1000, 2000, 3000, 4000, 5000$ and the dashed line is a fitted linear line obtained by least square distance rule. The solid line is the result from Multi-Gaussian model.
which means in linear scale, the fitting is $\tau_1 = 0.01888Re$. The simulation results are compared to the first bifurcation time $\tau_1 = 0.064\Gamma/\nu$ as predicted by the multi-Gaussian model. While the first bifurcation in the simulations happens earlier than in the model, both the Navier-Stokes simulations and model indicate that $\tau_1$ is linearly dependent on $Re$ (or $\Gamma/\nu$).

Figure 3.57: Streamlines of relative velocity field for the fixed equilibrium in rotating frame at various instants.
**Relative velocity field**  Similar to the previous cases, the governing equations for the relative velocity field in rotating frame are given by

\[
\dot{\zeta} = \dot{\theta}\eta + \frac{\eta}{2\pi(\zeta^2 + \eta^2)} \left[ 1 - \exp\left( -\frac{\zeta^2 + \eta^2}{4\tau} \right) \right] \\
- \frac{\eta}{\pi( (\zeta + 1)^2 + \eta^2)} \left\{ 1 - \exp\left( -\frac{(\zeta + 1)^2 + \eta^2}{4\tau} \right) \right\} \\
- \frac{\eta}{2\pi( (\zeta - 1)^2 + \eta^2)} \left\{ 1 - \exp\left( -\frac{(\zeta - 1)^2 + \eta^2}{4\tau} \right) \right\},
\]

\[
\dot{\eta} = -\dot{\theta}\zeta - \frac{\zeta}{2\pi(\zeta^2 + \eta^2)} \left[ 1 - \exp\left( -\frac{\zeta^2 + \eta^2}{4\tau} \right) \right] \\
+ \frac{\zeta + 1}{\pi((\zeta + 1)^2 + \eta^2)} \left\{ 1 - \exp\left( -\frac{(\zeta + 1)^2 + \eta^2}{4\tau} \right) \right\} \\
+ \frac{\zeta - 1}{\pi((\zeta - 1)^2 + \eta^2)} \left\{ 1 - \exp\left( -\frac{(\zeta - 1)^2 + \eta^2}{4\tau} \right) \right\},
\]

(3.27)

The associated relative velocity field is plotted in Figure 3.57. Again, the fixed points of the system are obtained by setting the right-hand-sides of above equations to zero. Due to symmetry, fixed points must lie on \(\zeta\) and \(\eta\) axis. On \(\eta\) axis, governing equation for \(\eta\) component of fixed point is

\[
-\dot{\theta}\eta - \frac{1}{2\pi\eta} \left[ 1 - \exp\left( -\frac{\eta^2}{4\tau} \right) \right] + \frac{2\eta}{\pi(\eta^2 + 1)} \left[ 1 - \exp\left( -\frac{\eta^2 + 1}{4\tau} \right) \right] = 0.
\]

(3.28)

The solutions of fixed points are the origin, \((0, \pm \eta_{f1}^*)\) and \((0, \pm \eta_{f2}^*)\). Figure 3.58(a) shows \(\pm \eta_{f1}^*\) as functions of time. One can see that \((0, \pm \eta_{f1}^*)\) start from \((0, \pm \sqrt{3}/3)\) then collapse at the origin at a finite time \(\tau_{1}^* \approx 0.064\), which is the first bifurcation time of the relative velocity field. Note that \(\tau_{1}^* = \tau_{1}\), i.e. the first bifurcation times
of absolute velocity field and relative velocity field are the same. Figure 3.58(b)

Figure 3.58: Vertical components of \((0, \pm \eta^*_{f1})\) and \((0, \pm \eta^*_{f2})\) as functions of time. The pair \((0, \pm \eta^*_{f1})\) start from \((0, \pm \sqrt{3}/3)\) then collapse at \((0, 0)\) at bifurcation time \(\tau^*_{f1} \approx 0.064\). And the pair \((0, \pm \eta^*_{f2})\) start from \((0, \pm \infty)\) and eventually converge to \((0, \pm \sqrt{11}/3)\) as \(\tau \to \infty\).

shows \(\pm \eta^*_{f2}\), which start from \(\pm \infty\) and eventually converge to their asymptotic values. We can analytically calculate \(\eta^*_{f2}|_{\tau \to \infty}\) by doing Taylor expansion of (3.28) as \(\tau \to \infty\),

\[
- \dot{\eta} - \frac{1}{2 \pi \eta} \left[ 1 - \exp \left( -\frac{\eta^2}{4 \tau} \right) \right] + \frac{2 \eta}{\pi (\eta^2 + 1)} \left[ 1 - \exp \left( -\frac{\eta^2 + 1}{4 \tau} \right) \right] \\
= - \frac{1}{2 \pi \eta^*_{f2}} \left[ \left( 1 - \frac{1}{4 \tau} + \frac{1}{32 \tau^2} \right) - \left( 1 - \frac{1}{\tau} + \frac{1}{2 \tau^2} \right) + O \left( \frac{1}{\tau^3} \right) \right] \\
- \frac{1}{2 \pi \eta^*_{f2}} \left[ 1 - \left( 1 - \frac{\eta^*_{f2}^2}{4 \tau} + \frac{\eta^*_{f2}^4}{32 \tau^2} \right) + O \left( \frac{1}{\tau^3} \right) \right] \\
+ \frac{2}{\pi (\eta^*_{f2}^2 + 1)} \left[ 1 - \left( 1 - \frac{\eta^*_{f2}^2 + 1}{4 \tau} + \frac{(\eta^*_{f2}^2 + 1)^2}{32 \tau^2} \right) + O \left( \frac{\eta^*_{f2}^6}{\tau^3} \right) \right] \\
\approx \frac{11 - 3 \eta^*_{f2}^2}{64 \pi \tau^2} = 0.
\]
Therefore, $\eta_{f2}|_{\tau \to \infty} = \sqrt{11/3} \approx 1.9$, which agrees with the numerical result. On the $\zeta$ axis, governing equation for the fixed points is

$$-\dot{\theta}\zeta - \frac{1}{2\pi\zeta} \left[ 1 - \exp\left( -\frac{\zeta^2}{4\tau} \right) \right] + \frac{2}{\pi(\zeta+1)} \left[ 1 - \exp\left( -\frac{(\zeta+1)^2}{4\tau} \right) \right] + \frac{2}{\pi(\zeta-1)} \left[ 1 - \exp\left( -\frac{(\zeta-1)^2}{4\tau} \right) \right] = 0.$$ 

The solutions are $(0,0)$, $(\pm 1, 0)$ and another pair $(\pm \zeta_f^*, 0)$ whose horizontal components are shown in Figure 3.59. One can see that $\pm \zeta_f^*$ start from $\pm \infty$ and reach $\pm 1$ at $\tau_3^* \approx 0.762$, which is the third bifurcation time, and eventually collapse at 0 in finite time $\tau_5^* \approx 0.818$, which is the fifth and final bifurcation time. We will show the other bifurcations in details later. In summary, the fixed points of the fixed equilibrium in rotating frame are: before $\tau_1^*$, the fixed points are $(0,0)$, $(\pm 1,0)$, $(\pm \zeta_f^*, 0)$, $(0,\pm \eta_{f1}^*)$, and $(0,\pm \eta_{f2}^*)$. After $\tau_1^*$ and before $\tau_5^*$, the fixed points are: $(0,0)$, $(\pm 1,0)$, $(\pm \zeta_f^*, 0)$, and $(0,\pm \eta_{f2}^*)$. After $\tau_5^*$, the fixed points are: $(0,0)$, $(\pm 1,0)$, and $(0,\pm \eta_{f2}^*)$.

![Figure 3.59: Horizontal components of $(\pm \zeta_f^*, 0)$. This pair $(\pm \zeta_f^*)$ reaches $(\pm 1, 0)$ at bifurcation time $\tau_3^* \approx 0.762$, and collapse at $(0,0)$ at bifurcation time $\tau_5^* \approx 0.818.\]
To obtain the elliptic/hyperbolic characters of the fixed points, following the similar procedure as before, we linearize (3.27) around the fixed points. As a result, linearized equations can be expressed in a common matrix form

\[
\begin{pmatrix}
\delta \dot{\zeta} \\
\delta \dot{\eta}
\end{pmatrix} = 
\begin{bmatrix}
0 & C_{12} \\
C_{21} & 0
\end{bmatrix}
\begin{pmatrix}
\delta \zeta \\
\delta \eta
\end{pmatrix},
\]

where \( C_{12} \) and \( C_{21} \) are different for each fixed point. Eigenvalues associated with the fixed points are \( \lambda = \pm \sqrt{C_{12}C_{21}} \).

For the origin, \( C_{12} \) and \( C_{21} \) are given by

\[
C_{12} = \dot{\theta} + \frac{1}{2 \pi \tau} - \frac{2}{\pi} \left[ 1 - \exp \left( -\frac{1}{4 \tau} \right) \right],
\]

\[
C_{21} = -\dot{\theta} - \frac{1}{8 \pi \tau} \left[ 1 - 8 \exp \left( -\frac{1}{4 \tau} \right) \right] - \frac{2}{\pi} \left[ 1 - \exp \left( -\frac{1}{4 \tau} \right) \right].
\]

Real and imaginary parts of the eigenvalues associated with the origin are plotted in Figure 3.60. One can see that before \( \tau_1^* \), the origin is an elliptic point; between \( \tau_1^* \)

\[
\begin{array}{c}
\text{Real} \\
\text{Imaginary}
\end{array}
\]

Figure 3.60: Real and imaginary parts of the eigenvalues of the origin for the fixed equilibrium. The origin is initially an elliptic point, after bifurcation time \( \tau_1^* \approx 0.064 \) it becomes hyperbolic, then after \( \tau_5^* \approx 0.818 \) it becomes an elliptic point again.
and $\tau_5^*$, it becomes a hyperbolic point; after $\tau_5^*$, it becomes an elliptic point again.

For $(\pm 1, 0)$, $C_{12}$ and $C_{21}$ are given by

$$C_{12} = \dot{\theta} + \frac{1}{4\pi \tau} \left[ 1 - \exp \left( -\frac{1}{\tau} \right) + \exp \left( -\frac{1}{4\tau} \right) \right] - \frac{1}{2\pi} \left[ 2 - \exp \left( -\frac{1}{\tau} \right) - \exp \left( -\frac{1}{4\tau} \right) \right],$$

$$C_{21} = -\dot{\theta} + \frac{1}{4\pi \tau} \left[ 1 + 2 \exp \left( -\frac{1}{\tau} \right) - \exp \left( -\frac{1}{4\tau} \right) \right] + \frac{1}{4\pi} \left[ 1 + \exp \left( -\frac{1}{\tau} \right) - 2 \exp \left( -\frac{1}{4\tau} \right) \right].$$

Real and imaginary parts of the eigenvalues associated with $(\pm 1, 0)$ are plotted in Figure 3.61. Before $\tau_3^*$, $(\pm 1, 0)$ are elliptic points; after $\tau_3^*$, they become hyperbolic points.

![Figure 3.61: Real and imaginary parts of the eigenvalues of $(\pm 1, 0)$ for the fixed equilibrium. $(\pm b_0, 0)$ are initially elliptic points, after bifurcation time $\tau_3^* \approx 0.762$ they become hyperbolic points.](image-url)
For \((\pm \zeta^*_f, 0)\), \(C_{12}\) and \(C_{21}\) are given by

\[
C_{12} = \dot{\theta} + \frac{1}{2\pi \zeta^*_f} \left[ 1 - \exp \left( -\frac{\zeta^*_f^2}{4\tau} \right) \right] - \frac{1}{\pi (\zeta^*_f + 1)^2} \left[ 1 - \exp \left( -\frac{(\zeta^*_f + 1)^2}{4\tau} \right) \right] \\
- \frac{1}{\pi (\zeta^*_f - 1)^2} \left[ 1 - \exp \left( -\frac{(\zeta^*_f - 1)^2}{4\tau} \right) \right],
\]

\[
C_{21} = -\dot{\theta} + \frac{1}{2\pi \zeta^*_f} \left[ 1 - \exp \left( -\frac{\zeta^*_f^2}{4\tau} \right) \right] - \frac{1}{\pi (\zeta^*_f + 1)^2} \left[ 1 - \exp \left( -\frac{(\zeta^*_f + 1)^2}{4\tau} \right) \right] \\
- \frac{1}{\pi (\zeta^*_f - 1)^2} \left[ 1 - \exp \left( -\frac{(\zeta^*_f - 1)^2}{4\tau} \right) \right] \\
+ \frac{1}{4\pi \tau} \left[ -\exp \left( -\frac{\zeta^*_f^2}{4\tau} \right) + 2 \exp \left( -\frac{(\zeta^*_f + 1)^2}{4\tau} \right) + 2 \exp \left( -\frac{(\zeta^*_f - 1)^2}{4\tau} \right) \right].
\]

Real and imaginary parts of the eigenvalues associated with \((\pm \zeta^*_f, 0)\) are plotted in Figure 3.62. Before \(\tau^*_3\), this pair are hyperbolic points. After \(\tau^*_3\) but before \(\tau^*_5\), they become elliptic points. After \(\tau^*_5\) they collapse at the origin, and remain elliptic points.
For \((0, \pm \eta^{*}_{fi})\), \(i = 1, 2\), \(C_{12}\) and \(C_{21}\) are given by the same form

\[
C_{12} = \dot{\theta} + \frac{1}{2\pi \eta^{*}_{fi}} \left[ -1 + \left( 1 + \frac{\eta^{*2}_{fi}}{2\tau} \right) \exp \left( \frac{-\eta^{*2}_{fi}}{4\tau} \right) \right] - \frac{2}{\pi (1 + \eta^{*2}_{fi})} \left[ \frac{1 - \eta^{*2}_{fi}}{1 + \eta^{*2}_{fi}} + \left( \frac{\eta^{*2}_{fi}}{2\tau} - \frac{1 - \eta^{*2}_{fi}}{1 + \eta^{*2}_{fi}} \right) \exp \left( \frac{-\eta^{*2}_{fi} + 1}{4\tau} \right) \right],
\]

\[
C_{21} = -\dot{\theta} - \frac{1}{2\pi \eta^{*2}_{fi}} \left[ 1 - \exp \left( \frac{-\eta^{*2}_{fi}}{4\tau} \right) \right] + \frac{2}{\pi (1 + \eta^{*2}_{fi})} \left[ \frac{1 - \eta^{*2}_{fi}}{1 + \eta^{*2}_{fi}} + \left( \frac{1}{2\tau} + \frac{1 - \eta^{*2}_{fi}}{1 + \eta^{*2}_{fi}} \right) \exp \left( \frac{-\eta^{*2}_{fi} + 1}{4\tau} \right) \right].
\]

Real and imaginary parts of the eigenvalues associated with \((0, \pm \eta^{*}_{fi})\) are plotted in Figure 3.63(a). This pair is always hyperbolic points. After \(\tau^{*}_{1}\), \((0, \pm \eta^{*}_{fi})\) coincide with the origin, and this part of eigenvalues are identical to that of the origin, i.e. also hyperbolic points. Real and imaginary parts of eigenvalues associated with \((0, \pm \eta^{*}_{f2})\) are plotted in Figure 3.63(b). They are elliptic points for all time.

Figure 3.63: Real and imaginary parts of the eigenvalues of (a) \((0, \pm \eta^{*}_{f1})\) and (b) \((0, \pm \eta^{*}_{f2})\) for the fixed equilibrium. Before \(\tau^{*}_{1}\), \((0, \pm \eta^{*}_{fi})\) are hyperbolic points. At \(\tau^{*}_{1}\), they collapse at origin. After \(\tau^{*}_{1}\), this part of the plot is the same with that of the origin, still hyperbolic points. The pair \((0, \pm \eta^{*}_{f2})\) are always elliptic points.
Figure 3.64 shows the fixed points and separatrices at various instants. At $\tau = 0$, the separatrices are the same with Figure 3.51(a) because initially the rotation rate is zero, hence relative velocity field is the same with absolute field. The first bifurcation in the streamline topology is due to the same mechanism explained in the absolute velocity field and takes place at the same time $\tau_1^* = \tau_1 \approx 0.064$, see Figure 3.64(c). The second bifurcation does not coincide in time with the second bifurcation identified in absolute velocity field, that is, $\tau_2^* \approx 0.6236 \neq \tau_2$. It is associated with a change in the streamline topology caused by a collapse of the separatrices associated with the hyperbolic pair $(\pm \zeta_f^*, 0)$ onto the separatrices of the now hyperbolic point at the origin, see Figure 3.64(e). The third bifurcation occurs at $\tau_3^* \approx 0.762$ when the hyperbolic points at $(\pm \zeta_f^*, 0)$ collide with the elliptic points at $(\pm 1, 0)$, respectively, causing them to change to hyperbolic points, see Figure 3.64(g). After the third bifurcation, one still has two pairs $(\pm \zeta_f^*, 0)$ and $(\pm 1, 0)$ of stagnation points on the $\zeta$ axis but with exchanged hyperbolic/elliptic characters. The fourth bifurcation takes place at $\tau_4^* \approx 0.802$ due yet to another collapse of the separatrices of the hyperbolic point at the origin with the separatrices at the now hyperbolic points at $(\pm 1, 0)$, see Figure 3.64(i). The fifth bifurcation takes place at $\tau_5^* \approx 0.818$ when the now elliptic pair $(\pm \zeta_h^*, 0)$ collides with the hyperbolic origin causing it to turn into an elliptic point, see Figure 3.64(k). This bifurcation sequence turns out to be crucial in dictating the time evolution of the vorticity field which we visualize using colored passive tracers as commonly done in experimental and computational fluid mechanics (see, for example, [105]).
Figure 3.64: Evolution of the separatrices of the relative velocity field. Instantaneous hyperbolic points are at intersections of separatrices and elliptic points are represented by circles. The outside separatrices and elliptic points in (b) and (c) are plotted out of scale. See context for details.

The homotopic equivalences of the original separatrices are plotted in Figure 3.65. The first state corresponds to $\tau = 0$. Note the outer separatrices only appear when $\tau > 0$. The bifurcation states are depicted in boxes. All other states
show the transitions between bifurcations. The last state persists as \( \tau \to \infty \). Interestingly, some states reappear in later times, and the difference between each appearance is the relative positions of hyperbolic points and vortex centers.

**Passive particle evolution** We seed the flow at time \( \tau = 0 \) with passive tracers of four different colors as shown in Figure 3.66(a) to distinguish the initial four fluid regions, namely, the three regions around the vortices bounded by the separatrix (seeded with red, blue and green particles, respectively) and the fourth region (seeded with yellow particles) bounded by the separatrix and the bound at infinity. We let the passive tracers be advected by the fluid velocity field given in (3.27). Snapshots of the passive tracers at six distinct instants in time are depicted in Figure 3.66. As time evolves, the location of the stagnation points and the associated separatrices change. Due to incompressibility, the particles initially in the region around the middle vortex (blue color) “leak” along the unstable branch of separatrices associated with the instantaneous hyperbolic points \((0, \pm \eta f_2)\). At \( \tau_1^* \), Figure 3.66(b) shows that all the particles are squeezed out of the middle region. Meanwhile as time progresses, the fluid particles in yellow begin to form lobes that

![Diagram](attachment:diagram.png)

**Figure 3.65:** Homotopic equivalences of the separatrices for fixed equilibrium.
stretch at a finite distance away from the initial location of the vortices, see Figure 3.66(c). Qualitatively, the passive tracers in Figure 3.66(c) indicate a vorticity field similar to that obtained from the Navier-Stokes simulation in Figure 3.53(d) (modulo the rigid rotation of the whole structure). The formation of these lobes cannot be explained based on the analysis of the streamline patterns in absolute velocity field. Indeed, the formation of these lobes is initiated when the yellow passive tracers encounter the separatrices associated with the hyperbolic points of the relative velocity field (3.27) \( \pm \zeta^*_f, 0 \) that appear from infinity and move towards the origin along the \( \zeta \) axis (see Figure 3.59). The lobes then stretch and rotate around the elliptic points \( (0, \pm \eta^*_f, 0) \) that appear from infinity and converge to a finite distance away from the origin (see Figure 3.58(b)). Eventually, the passive particles initially placed in the regions around the point vortices, whose detailed evolution is also dictated by the sequence of bifurcations described in Figure 3.64, join the large lobes as well and begin to stretch and rotate at a finite distance away from the initial vortex configuration, see Figure 3.66(d)-(f). After the last bifurcation in Figure 3.64(k), all the passive particles continue to rotate as shown in Figure 3.66(f) while spreading further and further as time evolves under the effect of viscosity. We emphasize that this interesting dynamics of the passive particles, which in turn indicates the evolution of the vorticity field, cannot be explained based solely on the analysis of the streamlines of the absolute velocity field. In addition, because of the detailed and delicate nature of the full series of topological bifurcations that occur,
to capture all but the first of these in a DNS would require considerable further effort and is beyond the scope of the current work.

(a) \( \tau = 0 \)  
(b) \( \tau = \tau^* \approx 0.064 \)  
(c) \( \tau = 0.344 \)  
(d) \( \tau = 0.48 \)  
(e) \( \tau = 0.62 \)  
(f) \( \tau = 1 \)

Figure 3.66: Colored passive tracers advected by the relative velocity field and depicted in the frame rotating with the vortex structure. As time evolves, the passive tracers stretch and mix forming large lobes at a finite distance from the initial location of the vortex structure. The separatrices of the relative velocity field are superimposed in black at various instants in time.
Chapter 4

Discussion and Conclusions

The redistribution (inviscid) and diffusion (viscous) of Dirac delta initial distributions of vorticity, although configuration independent for sufficiently long timescales (for non-zero circulation flow field), is highly dependent on the initial positions and strengths of the point vortices on short and intermediate timescales. These are typically the timescales in which much of the important mixing, transport, and redistribution of vorticity is achieved in many settings. Greengard’s paper [33] notwithstanding pointing out that the types of models based on advection and core diffusion are not exact solutions of the Navier-Stokes equations, these ideas are exceptionally useful in getting a handle on some of the important dynamical mechanisms that occur during the evolution towards the ultimate Lamb-Oseen state. In fact, one contribution of the current work is to further quantify and understand the limitations of “core growing” type models as diagnostic tools for understanding more and more complex flows and to point out some of the delicate
issues in comparing a DNS with these models [40]. Not surprisingly, core spreading type models are also useful as starting points for more sophisticated numerical methods which systematically exploit some of the main features [93, 22]. Also, see Blobflow, an open source vortex method package developed by Rossi, available at http://www.math.udel.edu/~rossi/BlobFlow as of June 2011. A numerical work closely related to this work is a multi-moment vortex method developed by Uminsky et al. [102], in which they used Hermite expansion to approximate the Navier-Stokes solution, and the terms in the expansion correspond to the moments of the flow. The multi-Gaussian model can be viewed as the leader term in this expansion, and one can include higher order terms to account for the stretching of the vorticity that is not present in the model. This will be addressed in a forthcoming study.

In inertial frame, the systems evolve towards Lamb-Oseen vortex when the net circulation in the flow field is nonzero, as shown in Section 3.1, 3.2 and 3.4. But the ultimate form is still unclear in a zero circulation field such as Section 3.3. The streamline patterns associated with the absolute velocity fields in the nonzero cases undergo clear sequences of topological bifurcations. The homotopic equivalences of topological change of separatrices associated with the absolute velocity field for the nonzero cases are replotted in Figure 4.1. The vortex tripole does not converge to a Gaussian vortex, yet, multi-Gaussian model reveals the bifurcation in the relative velocity field, therefore defines (possibly) the only timescale in the evolution sequence.
The presence of viscosity causes the structures to rotate unsteadily. In the first three cases where the initial conditions are inviscid relative equilibria, the rotation rates in viscous flow decay to zero gradually due to viscosity. In the last case, the initial condition (fixed equilibrium) is known to be unstable in inviscid fluid, as pointed out comprehensively in [99, 6]. The presence of viscosity immediately “triggers” the underlying instability of the equilibrium, causing the structure to rotate unsteadily, which we refer to as a “viscously-induced rotation”, and, of course, eventually the velocity field also diffuses to zero everywhere.

The rotating frame is essential to the understanding of viscous vorticity evolution, yet the choice of rotation rate and the reason to move into this frame are not always well justified. In the cases we studied, since each structure of vortex centers
Figure 4.2: Homotopic equivalences of relative velocity field separatrices for (a) asymmetric co-rotating pair, (b) symmetric co-rotating pair, (c) tripole and (d) fixed equilibrium.

does not change shape, angular velocity of the structure $\dot{\theta}$ is a natural choice of rotation rate, and the analogy between the velocity field and a Rankine vortex justifies the physical meaning of subtracting a rigid body rotation from the (rotated)
absolute velocity field, see Figure 3.9. Indeed, although the passive particle evolution is a result from integrating the absolute velocity field, it is obvious that the relative velocity fields are much more relevant to the particle evolution than the absolute velocity fields (the homotopic equivalences of the relative fields are replotted in Figure 4.2). The reason is the following: as noted by Velasco Fuentes [27], the Lagrangian dynamics of the passive particle trajectories do not always follow the *time varying* Eulerian streamlines. In steady flows, both geometries are identical. The slower the Eulerian velocity field is varying, the less difference between the two [36, 64]. In the cases studied, the absolute velocity field is a fast varying Eulerian field, which is both rotating and evolving, and the relative velocity field is a much slower varying Eulerian field since it is a result of subtracting the fast varying part (rigid body rotation) off the absolute field. One can derive the expression of vorticity in this rotating frame as following: by definition,  \( \hat{\omega}(\xi, \tau) = \nabla_\xi \times \dot{\xi} \), where \( \hat{\cdot} \) is used to distinguish from the vorticity in inertial frame \( \omega(z, \tau) \), and the subscript
\( \xi \) indicates that curl is taken in rotating frame. Substituting \( \dot{\xi} = R^T \dot{z} - R^T \dot{R} R^T z \) into vorticity and using the chain rule, one arrives at the vorticity in rotating frame

\[
\hat{\omega}(\xi, \tau) = \nabla_\xi \times \dot{\xi} = \frac{\partial \dot{\eta}}{\partial \zeta} - \frac{\partial \dot{\zeta}}{\partial \eta} = \frac{\partial}{\partial \zeta}(-\dot{x} \sin \theta + \dot{y} \cos \theta - \dot{\theta} x \cos \theta - \dot{\theta} y \sin \theta) - \frac{\partial}{\partial \eta} (\dot{x} \cos \theta + \dot{y} \sin \theta - \dot{\theta} x \sin \theta + \dot{\theta} y \cos \theta) = \left( \cos \theta \frac{\partial}{\partial x} + \sin \theta \frac{\partial}{\partial y} \right) (-\dot{x} \sin \theta + \dot{y} \cos \theta - \dot{\theta} x \cos \theta - \dot{\theta} y \sin \theta) - \left( -\sin \theta \frac{\partial}{\partial x} + \cos \theta \frac{\partial}{\partial y} \right) (\dot{x} \cos \theta + \dot{y} \sin \theta - \dot{\theta} x \sin \theta + \dot{\theta} y \cos \theta) = \frac{\partial \dot{y}}{\partial x} - \frac{\partial \dot{x}}{\partial y} - 2 \dot{\theta}.
\]

Therefore,

\[
\hat{\omega}(\xi, \tau) = \omega(z, \tau) - 2 \dot{\theta}.
\] (4.1)

At any given position \( \xi \), the difference between \( \hat{\omega} \) and \( \omega \) at the corresponding \( z \) is \(-2 \dot{\theta}\), which does not depend on \( \xi \) or \( z \).

The topology patterns of symmetric co-rotating pair are identical to the last 8 states in the fixed equilibrium case. This is an evidence of the point made in Section 3.4: in the fixed equilibrium, the center vortex is weaker \((-\Gamma)\) comparing to the side vortices \((2\Gamma)\), therefore the side vortices are the “drivers” in the flow field since the center one diffuses much faster. Once the center vortex diffuses out, the remaining two vortices have the same strength, therefore the flow field is very similar to that in the symmetric co-rotating case.
Comparison between the symmetric and asymmetric vortex pairs gives insight of common and different features between symmetric and asymmetric configurations in general. Their rotation rates are the same since they have identical total circulation in the flow field. The absolute velocity fields are also similar as seen in Figure 3.4 and 3.23: both start with the separatrices in ∞-shape and eventually converge to a single Gaussian vortex. However, the asymmetry is evident from Figure 3.23, the right ring of the ∞-shape separatrix shrinks to zero first, and the left ring remains, rather than the simultaneous disappearance in the symmetric case. Furthermore, the difference is much more obvious in the relative velocity field, and the topologies of the two cases are different even when $\tau \to \infty$.

The results of the vortex pairs can be partially verified in comparison to the experimental results in previous studies, for example, Figure 13 in Cerretelli & Williamson [19] (shown in Figure 4.3), where they obtained separatrices topologically identical to Figure 3.15(a) and (h), but the intermediate states of the topological change were not explicitly available. For the asymmetric case, to our knowledge, no previous research has shown separatrices in the rotating frame other than the initial state. We present the \textit{complete} sequence based on the multi-Gaussian model in Figure 3.15 for the symmetric case and Figure 3.30 for the asymmetric case, and one can see that the bifurcations are very rich. People have studied the vortex pair merging process and determined the “onset” of merging, and categorized the interaction of the asymmetric pairs. In this work, we do \textit{not} attempt to quantify these phenomenon. Because the co-rotating pair (regardless of the strength ratio)
will always completely merge given enough time, and the merging process starts as soon as time is greater than 0 because the spreading of vorticity fields associated with the initial Dirac delta peaks is infinitely fast. However, we find the criterions obtained by previous studies in determining the onset and categorization extremely valuable in explaining the passive particle evolution, given its Lagrangian nature. One critical value in the symmetric vortex merger is the ratio between the core size $a$ and the distance between the two symmetric vortices $b$. As mentioned before, we define the core size to be the Gaussian core $a = \sqrt{4\nu t} \equiv \sqrt{4\tau}$. In Figure 3.17(b), $a = \sqrt{4 \times 0.0235} \approx 0.307$, the edge of core (not shown in the plot) touches the inner separatrix, which can be thought of as the majority of vorticity remains inside the inner core (inside the 1-sigma region), but the rest of the vorticity already enters the exchange band. After that, more vorticity will enter the exchange band, which can be visualized by the passive particles in Figure 3.17(c). If we define the distance between the vortices to be the distance between the vortex peaks, as shown in Figure 3.3(b), at $\tau = 0.0235$, the distance $b \approx b_0 = 1$. Therefore, the critical value is $(a/b)_{cr} \approx 0.307$, which, interestingly, is in the range of “onset” time determined.
by previous studies, but we associate this relation to the core size and separatrices in rotating frame. Similarly, in the asymmetric case, at $\tau = 0.0135$, the edge of the right core (which is the same size with the left core since the viscosity is the same) touches the inner separatrix, while the left core is still inside the inner separatrix due to asymmetry. At this time, some passive particles are already present in the exchange band due to our choice of seeding. In Trieling et al. [101] and Brandt & Nomura [16], the authors considered different strength ratios of asymmetric cases, and the trend is: when $\Gamma_L/\Gamma_R$ is far away from 1, the stronger vortex seems to remain intact while the weaker vortex wraps around the stronger one; when $\Gamma_L/\Gamma_R$ is close to 1, the two vortices both deform and merge. We expect the same trend in our model, and the underlying physics is the competition between the growing core size (independent of vortex strength ratio) and the shrinking of separatrices (depends on vortex strength ratio).

Figure 4.4: Four stages of merging process based on distance between symmetric vortices, from Cerretelli & Williamson [19].
Another important value in symmetric vortex mergers is the distance $b$ between the vortices. Cerretelli & Williamson [19] and Trieling et al. [101] found four stages of vortex merger based on the nondimensionalized distance $b/b_0$: first diffusive stage, convective stage, second diffusive stage, and merged diffusive stage, see Figure 4.4. Using the multi-Gaussian model, there are multiple aspects of the problem (as shown in different paragraphs in each section of the four cases), hence we have multiple candidates for the definition of $b$ that are mathematically sound: (i) distance between the elliptic points in inertial frame, (ii) distance between the elliptic points on the horizontal axis in rotating frame, and (iii) distance between vorticity peaks, see Figure 4.5. But, as argued before, candidate (i) is of no direct relevance of the vorticity evolution. And candidate (ii) is also not directly relevant, as can be seen in the passive particle evolution in Figure 3.17 after $\tau^*_2$. Candidate (iii) is the most relevant, as can be seen in Figure 4.5 and 4.4, but it does not
explain the second diffusive process. We will address the second diffusive process in a forthcoming work based on the passive particle evolution.

We finish Part I of this work by mentioning connections of this work in two other contexts. First, there is by now a growing body of work on calculating “time-dependent separatrices” in developing flows that goes under the name of “Lagrangian coherent structures” (LCS) [35, 37]. Certainly these tools are potentially useful for further elucidating the intermediate timescale dynamics associated with the evolution towards the Lamb-Oseen state, particularly for more complex initial patterns that perhaps start out as relative equilibria of the Euler equations.

Second, if one regards, the vorticity field as a probability density function associated, for example, with the positions of initial system of point vortices undergoing a random walk, there are meaningful interpretations of the models used in this paper that have been discussed most recently, for example, in [2, 3, 50]. While this interpretation has not been the main focus of our work, we do find it potentially ripe for future development.
Part II

Effects of Body Elasticity on Stability of Fish Motion
The study of swimming dynamics of fish has a long and fascinating history, and the first work addressing this problem can be traced back to the books by Borelli *De Motu Animalium (On the Movement of Animals)* published in the early 1680s. Recently, this field gains more interest due to the rise of researches in biological and bio-mechanic applications. Early efforts in developing models for fish swimming can be attributed to Taylor [100], Wu [111], Lighthill [61] and Newman & Wu [83].

In light of these works, researches emerged to study and implement the principle of shape change instead of using the traditional propellers on underwater vehicles, we mention, for example, the works of Mason & Burdick [69], Morgansen et al. [79], Kelly [48] and Radford [89]. Experimentally, recent studies have provided evidence that fish prefer to exploit and couple their shape change with the circulation in the surrounding fluid in order to reduce energy expenditure during locomotion, see the works of Müller [80], Liao et al. [59, 60] and Beal et al.[13]. For instance, in the work by Beal et al., a freshly dead fish is placed between the von Kármán vortex
street generated by an ambient flow past a D-shaped cylinder, and the experiments show that the dead fish can *passively* swim upstream between the vortices, a phenomenon they concluded primarily due to the passive reaction of body elasticity to the surrounding flow. A series of analytical and numerical models have been developed to understand the similar swimming mechanisms, see the numerical works of Eldredge [24, 112] and the analytical works of Kanso et al. [45] and Kanso [44], just to name a few. In the work of Kanso et al, a model of fish locomotion as an articulated body made of rigid links submerged in an ideal fluid was developed, and the result showed that even in an ideal fluid (vorticity is not present in the flow), the articulated body can still propel and steer itself by actively changing its shape.

Figure 5.1: Schematic of the “burst and coast” cycle in fish swimming, widely believed to reduce energy expenditure in locomotion (Figure inspired by Videler & Weihs [106]).

Fish seem to alternate between actively controlling their shape to propel and passively responding to the surrounding fluid to glide: a cycle of motion commonly regarded to as “burst and coast”, see Videler & Weihs [106]. We are interested in studying the stability of the coast motion of fish in this work. In general, the stability of fish swimming has also intrigued interests among biologist, physicists and bio-mechanics. Researches on stability have mainly focused on hydrostatic equilibrium and periodic motion, see, for example, Weihs [109, 110] and Webb &
Weihs [108]. Relatively less attention has been attributed to the coast stability. Some studies suggest that the coast motion seems to be intrinsically unstable [97] and fins seem to be the stabilizing mechanism at play, yet the drag predicted by these models was higher than expected [107, 14].

Stability and maneuverability are balanced among aquatic animals, highly maneuverable bodies tend to be unstable, as pointed out by Fish [25]. In the same work, it was also shown that circular objects are more stable in a fluid environment, while an elongated shaped bodies are intrinsically unstable. The stability of a single submerged ellipsoid (or ellipse in 2D) moving at a constant translational velocity along the major axis of symmetry is found to be unstable in Lamb [54], Marsden et al [68] and Leonard [58]. This seems to suggest that if one simply models the fish as a rigid elongated body, the coast motion is unstable, which is counter intuitive in nature. The rigid model does not account for the possibility of shape deformation and body elasticity. Inspired by the passive swimming of the dead fish due to its elasticity as shown in [13], we are particularly interested in asking the question: can the presence of shape change and body elasticity affect the stability of the coast motion? Interestingly, the answer is yes. We adopt the articulated body model in Kanso et al. [45] and keep the assumptions of perfect fluid and zero vorticity in the flow field. But, instead of actively controlling the shape of fish, we let the body freely interact with the surrounding fluid. Muscle elasticity is accounted for by the torsional springs at the joints between the rigid links. Our analysis predicts a range of parameter values (geometry and stiffness) for which the coast motion is stable.
Figure 5.2: Single ellipse in 2D (ellipsoid in 3D) submerged in perfect fluid is unstable when moving along its major axis of symmetry, i.e. unstable when $b < a$ ($b < a, c < a$ in 3D).

The organization of Part II of this work is as follows: in Chapter 6, we derive equations of motion for the body-fluid system using Newtonian and Lagrangian approaches for both hydrodynamically decoupled and coupled models in 2D and 3D. The resulting equations admit a family of relative equilibria corresponding to the coast motion. We investigate the linear stability of these equilibria for two-link and three-link models in 2D in Chapter 7, and three-link model in 3D in Chapter 8, as examples of the general $N$-link model. We find that for certain parameter values, these equilibria are linearly stable. The linear results are verified by nonlinear simulations. The stability results are summarized in Chapter 9.
Chapter 6

Equations of Motion

6.1 Fish model and kinematics

![Diagram of a three-link model of a fish in 3D]

Figure 6.1: Three-link model of fish in 3D: an articulated body made of three identical ellipsoids $B_i$, $i = 1, 2, 3$, with semi-axes $a, b$ and $c$, connected by hinge joints placed distance $l$ ($l > a$) away from the centers of the ellipsoids along their major axes. The joints are constrained so that the shape change only occurs in $\Sigma$, a plane parallel with the $(a, b)$ plane of all ellipsoids. Muscle elasticity is accounted for by torsional springs $k_i$ at the joints between $B_i$ and $B_{i+1}$. Shape is parametrized by the relative angles $\theta_i$, $i = 1, 2$. The articulated body is neutrally submerged in perfect fluid $F$.
In general, fish is modeled in 3D as a neutrally buoyant articulated body moving in an infinitely large volume of incompressible, inviscid and irrotational fluid $\mathcal{F}$ at rest at infinity. Let the body be made of $N$ identical rigid ellipsoids of uniform density $\rho$ (equal to that of the fluid) with semi-axes $a$, $b$ and $c$. The rigid links (or ellipsoids) are labeled $B_i$, $i = 1, \ldots, N$, and connected via $N - 1$ massless and frictionless hinge joints placed distance $l$ away from the centers of the ellipsoids along their major axes ($l > a$). Shape change of the articulated body is parametrized by the relative angles $\theta_i$ between $B_i$ and $B_{i+1}$. And the shape change is constrained by the hinge joints such that it only occurs in a plane $\Sigma$ that is parallel with the $(a, b)$ plane of all ellipsoids. Between the joints and tips of ellipsoids are massless virtual rods. Muscle elasticity of the fish is accounted for by torsional springs with stiffness constants $k_i$ between $B_i$ and $B_{i+1}$. An example of $N = 3$, i.e. a three-link fish model is depicted in Figure 6.1. Mass of each body is $m^B$, and the principle moment of inertia of each ellipsoid is $J^B$. The fish can also be simply modeled in 2D. The setup is similar to 3D, but instead of ellipsoids, the articulated body is now made of ellipses. The 2D model will be discussed in detail in later sections. The terms “ellipsoids” (or “ellipses” in 2D), “bodies” and “links” will be used interchangeably in this work.

We first describe the kinematics of fish in a coordinate independent way. It is convenient for studying fish swimming to introduce an inertial frame $\{e_k\}_{k=1,2,3}$ originated at $O$. Also, one can attach a body-fixed frame $\{b_k^i\}_{k=1,2,3}$ to the center of each ellipsoid $C_i$, where $b_1^i$, $b_2^i$ and $b_3^i$ are parallel to the $a$, $b$ and $c$ semi-axes of $B_i$. 

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respectively. One identifies the position and orientation of each submerged ellipsoid with $SE(3)$, the group of rigid body motion in $\mathbb{R}^3$. One element $(Q_i, x_i) \in SE(3)$ associated with $B_i$ consists of a vector $x_i \in \mathbb{R}^3$ that describes the position of mass center $C_i$ and a proper orthogonal tensor of rotation $Q_i \in SO(3)$ that describes the orientation of $B_i$. The transformation between the inertial frame and body frame is the following: if $X$ is any material point in $B_i$ expressed in its body-fixed frame, and $x$ is the same point expressed in inertial frame (in this work, variables in body-fixed frame will typically be denoted in capital letters, and in inertial frame in lower letters), the relation is given by

$$x = Q_i X + x_i .$$

(6.1)

Linear velocity of the mass center $C_i$ relative to origin $O$ expressed in inertial frame is given by

$$v_i = \frac{dx_i}{dt} \equiv \dot{x}_i ,$$

while that expressed in its body frame is given by $V_i$. The relation between these variables is given by

$$v_i = Q_i V_i .$$

(6.2)

Angular velocity of $B_i$ with respect to inertial frame but expressed in inertial frame is denoted by $\omega_i$, while that expressed in body frame is denoted by $\Omega_i$. Define the
map \( \hat{\cdot} : \mathbb{R}^3 \to \mathfrak{so}(3) \), such that \( \hat{\alpha} \beta = \alpha \times \beta \) for all \( \alpha, \beta \in \mathbb{R}^3 \). The angular velocity is given by the skew-symmetric tensor

\[
\hat{\omega}_i = \hat{Q}_i Q_i^T, \quad \text{or} \quad \hat{\Omega}_i = Q_i^T \dot{Q}_i,
\]

where \( \dot{Q}_i \equiv dQ_i/dt \), and \( Q_i^T \) indicates the transpose of \( Q_i \). Similar to the linear velocity, the relation between angular velocity expressed in inertial and body frames can be written as

\[
\omega_i = Q_i \Omega_i.
\]

The links are connected with each other by mechanical constraints from the hinge joints and hydrodynamical interaction via the fluid. We first examine the mechanical constraints. Position of the \( i^{th} \) joint \( K_i \) expressed in inertial frame is denoted \( x_{K_i} \). From (6.1), \( x_{K_i} \) can be written in terms of positions and orientations of the links on both sides of the joint, that is to say,

\[
x_{K_i} = Q_i X_{K_i}^i + x_i, \quad x_{K_i} = Q_{i+1} X_{K_i}^{i+1} + x_{i+1},
\]

where \( X_{K_i}^i \) and \( X_{K_i}^{i+1} \) denote the position of \( K_i \) (which is the subscript) expressed in the \( B_i \)-fixed and \( B_{i+1} \)-fixed frames (which is the superscript), respectively. Therefore,

\[
Q_i X_{K_i}^i + x_i = Q_{i+1} X_{K_i}^{i+1} + x_{i+1}.
\]
Take time derivative of above equation and also consider (6.3), one has

\[
\dot{Q}_i \dot{X}_i^{i+1} + Q_i \ddot{X}_i^{i+1} + \dot{x}_i = \dot{Q}_{i+1} \dot{X}_{K_i}^{i+1} + Q_{i+1} \ddot{X}_{K_i}^{i+1} + \dot{x}_{i+1}
\]

\[
Q_i \dot{\Omega}_i X_i^{i+1} + Q_i V_i = Q_{i+1} \dot{\Omega}_{i+1} X_{K_i}^{i+1} + Q_{i+1} V_{i+1}
\]

\[
\Omega_i \times X_i^{i+1} + V_i = Q_i^T Q_{i+1} (\Omega_{i+1} \times X_{K_i}^{i+1} + V_{i+1})
\]

since \( \dot{X}_{K_i} = 0 \) no matter in which body frame it is expressed. The relative orientation of \( B_{i+1} \) with respect to \( B_i \) is denoted by \( R_{i+1,i} \). Mathematically, this relative orientation is given by

\[
R_{i+1,i} = Q_i^T Q_{i+1} .
\]

(6.5)

Physically, \( R_{i+1,i} \) transforms from the \( B_{i+1} \)-fixed frame to the \( B_i \)-fixed frame. Obviously,

\[
R_{i+1,i}^T = R_{i+1,i}^{-1} = R_{i,i+1} .
\]

Take time derivative of (6.5),

\[
\dot{R}_{i+1,i} = \dot{Q}_i^T Q_{i+1} + Q_i^T \dot{Q}_{i+1}
\]

\[
= -\dot{\Omega}_i Q_i^T Q_{i+1} + Q_i^T Q_{i+1} \dot{\Omega}_{i+1}
\]

\[
= -\dot{\Omega}_i R_{i+1,i} + R_{i+1,i} \dot{\Omega}_{i+1} .
\]

(6.6)

Therefore, the mechanical constraint in terms of velocities takes the form

\[
\Omega_i \times X_i^{i+1} + V_i = R_{i+1,i} (\Omega_{i+1} \times X_{K_i}^{i+1} + V_{i+1})
\]

(6.7)
6.2 Equations of motion for a single ellipsoid

Before deriving the equations for the articulated body that consists of \( N \) links, we first derive equations of motion for a single ellipsoid \( B \) neutrally submerged in a perfect fluid \( F \) with no external force or torque. The orientation and position of \( B \) is given by \( Q \) and \( x \) with respect to the inertial frame \( \{ e_k \}_{k=1,2,3} \) originated at \( O \). Again, one can attach a body-fixed frame \( \{ b_k \}_{k=1,2,3} \) to the mass center \( C \).

Linear and angular velocities of the body in inertial frame are denoted by \( v \) and \( \omega \), respectively, while that expressed in body-fixed frame are denoted by \( V \) and \( \Omega \).

**Reduced kinetic energy** Since only one body exists in the fluid and the body is neutrally buoyant, potential energy of the system is always zero. Kinetic energy of the body-fluid system is equal to kinetic energy of the body \( T_B \) plus kinetic energy of the fluid \( T_F \), namely,

\[
T = T_B + T_F.
\]
Kinetic energy of the body $T_B$ can be written in terms of body-frame velocities

$$T_B = \frac{1}{2} (V \cdot m^B V + \Omega \cdot J^B \Omega). \quad (6.8)$$

And kinetic energy of the fluid $T_F$ is given in spatial representation by

$$T_F = \frac{1}{2} \iiint_F \rho \|u\|^2 \, dv, \quad (6.9)$$

where the readers are reminded that the density $\rho$ of the fluid is identical to that of the body since it is neutrally buoyant, $u$ is the velocity field of the fluid, and $dv$ is a standard volume element in $\mathbb{R}^3$. For a perfect fluid, the fluid velocity can be written as the gradient of a scalar potential function

$$u = \nabla \phi. \quad (6.10)$$

Due to incompressibility, the potential $\phi$ is a solution of Laplace’s equation,

$$\nabla \cdot u = 0 \quad \Rightarrow \quad \nabla \cdot \nabla \phi = \Delta \phi = 0,$$

subject to the impermeability (non-penetration) boundary condition on the surface of $B$, and a proper decay at infinity, i.e.

$$\begin{cases} 
\nabla \phi \cdot n = (V + \Omega \times X) \cdot n & \text{on } \partial B, \\
\n\nabla \phi = 0 & \text{at } \infty. 
\end{cases} \quad (6.11)$$
Here, \( \mathbf{n} \) is a unit vector normal to the boundary \( \partial \mathcal{B} \) pointing outward (into the fluid), \( \mathbf{X} \) is the position vector of a material point on \( \partial \mathcal{B} \) expressed in body frame, and the term \( (\mathbf{V} + \mathbf{\Omega} \times \mathbf{X}) \cdot \mathbf{n} \) is the normal component in the \( \mathbf{n} \) direction of the velocity of this material point. Now, invoke Green’s theorem to get

\[
T_F = \frac{1}{2} \rho \iiint_{\mathcal{F}} \nabla \phi \cdot \nabla \phi \, dv \\
= \frac{1}{2} \rho \iiint_{\mathcal{F}} \left[ \text{div}(\phi \nabla \phi) - \phi \Delta \phi \right] \, dv \\
= \frac{1}{2} \rho \iiint_{\mathcal{F}} \text{div}(\phi \nabla \phi) \, dv \\
= \frac{1}{2} \rho \left( \iint_{\mathcal{F}} - \iint_{\partial \mathcal{B}} \right) \phi \nabla \phi \cdot \mathbf{n} \, da \\
= -\frac{1}{2} \rho \iint_{\partial \mathcal{B}} \phi \nabla \phi \cdot \mathbf{n} \, da,
\]

where the boundary condition at infinity and the identity

\[
\text{div}(\phi \nabla \phi) = \nabla \phi \cdot \nabla \phi + \phi \Delta \phi
\]

is already taken into consideration, and \( da \) is a standard area element in \( \mathbb{R}^2 \). The integration reduces from a space integration over the whole fluid domain \( \mathcal{F} \) in \( \mathbb{R}^3 \) to a surface integration only over the boundary \( \partial \mathcal{B} \) in \( \mathbb{R}^2 \). In order to solve \( \phi \) and \( \nabla \phi \) appeared in (6.12), following Lamb [54], one can write \( \phi \) in Kirchhoff form

\[
\phi = \mathbf{V} \cdot \mathbf{\varphi} + \mathbf{\Omega} \cdot \mathbf{\chi},
\]

(6.13)
where $\varphi$ is called the *translational velocity potential* and $\chi$ is called the *rotational velocity potential*. Clearly, $\varphi$ and $\chi$ are dimensionally inhomogeneous; their entries have the units of length and length squared, respectively. Due to linearity, $\varphi$ and $\chi$ are also solutions of Laplace’s equation

$$\Delta \varphi = 0, \quad \Delta \chi = 0.$$  \hspace{1cm} (6.14)

The boundary conditions of these potentials can be obtained by substituting (6.13) into the boundary conditions in (6.11), which gives

$$\nabla \varphi \cdot n \equiv \frac{\partial \varphi}{\partial n} = n, \quad \nabla \chi \cdot n \equiv \frac{\partial \chi}{\partial n} = X \times n,$$  \hspace{1cm} (6.15)

on the boundary $\partial B$, and zero at infinity. Therefore the velocity potentials $\varphi$ and $\chi$ depend *solely* on the shape configuration of the body (via $n$ and $X$ on the boundary), *not* on its motion. In general, the potential function $\phi$ or velocity potentials associated with an arbitrary shaped body does not have closed form solution. One has to solve $\phi$ numerically using, for example, panel method in order to enforce the boundary condition. However, for the symmetric shapes like a single ellipsoid, analytical form of $\phi$ has been derived and one is referred to Lamb [54] for details. In general, substitute (6.13) into (6.12), the kinetic energy of the fluid can be expressed as

$$T_f = \frac{1}{2} (V \cdot \Theta^{\varphi^2} V + V \cdot \Theta^{\varphi\chi} \Omega + \Omega \cdot \Theta^{\varphi^2} V + \Omega \cdot \Theta^{\chi^2} \Omega),$$
where the “mass matrices” $\Theta$ are determined by the configuration of the body, for example, $\Theta^{\phi \chi}$ is given by

$$
\Theta^{\phi \chi} = -\frac{1}{2} \rho \int_{\partial B} \left( \phi \otimes \frac{\partial \chi}{\partial n} + \frac{\partial \phi}{\partial n} \otimes \chi \right) \, da,
$$

where the tensor product $\otimes$ is defined through its operation on $\gamma$ such that $(\alpha \otimes \beta)\gamma = (\beta \cdot \gamma)\alpha$ for arbitrary $\alpha, \beta, \gamma \in \mathbb{R}^3$. Denote the linear and angular velocities in body frame in a compact form as

$$
W = \begin{pmatrix} V \\ \Omega \end{pmatrix}.
$$

The kinetic energy of the fluid can also be written in a compact form

$$
T_F = \frac{1}{2} W \cdot M^F W, \quad M^F = \begin{bmatrix} \Theta^{\phi \phi} & \Theta^{\phi \chi} \\ \Theta^{\chi \phi} & \Theta^{\chi \chi} \end{bmatrix},
$$

where $M^F$ is a $6 \times 6$ matrix whose components depend solely on the configuration of the body, and is generally referred to as the added mass or added inertia matrix.

One can rewrite the kinetic energy of the body (6.8) in a similar matrix form

$$
T_B = \frac{1}{2} W \cdot M^B W, \quad M^B = \begin{bmatrix} m^B I_{3 \times 3} & 0 \\ 0 & J^B \end{bmatrix}.
$$
where $I_{3\times3}$ is a $3 \times 3$ identity matrix. Hence the kinetic energy the solid-fluid system can be written as

$$T = T_B + T_F = \frac{1}{2} \mathbf{W} \cdot \mathbf{M} \mathbf{W},$$

(6.16)

where $\mathbf{M} = \mathbf{M}^B + \mathbf{M}^F$ is referred to as the total mass of the solid-fluid system. One can see that the system is reduced to being represented by the body velocity $\mathbf{W}$ only, and fluid velocity field $\mathbf{u}$ does not appear in (6.16), though the effect of the perfect fluid is “encoded” in the added mass matrix. To this end, one can intuitively think of the solid-fluid system as an “augmented” body in the space, and the “augmentation” part is given by the added mass, which solely depends on the configuration of the body. This reduction greatly helps in the simplification of thinking process, yet caution must be in order here. Although there is an analogy between $\mathbf{M}^B$ and $\mathbf{M}^F$, they are fundamentally distinct. For example, unlike the body’s actually mass, the translational part of added mass may differ depending on the direction of the body motion.

For concreteness, we express the variables in component form. The position of mass center $C$ is given in inertial frame by $\mathbf{x} = xe_1 + ye_2 + ze_3$, or equivalently $\mathbf{x} = (x, y, z)$. The orientation $Q$ is a proper orthogonal rotation tensor, and is typically represented in Euler angles or Rodriguez formulation. Linear velocity can be written in inertial frame as $\mathbf{v} = v_x e_1 + v_y e_2 + v_z e_3 \leftrightarrow (v_x, v_y, v_z)$, while in body fixed frame as $\mathbf{V} = V_x b_1 + V_y b_2 + V_z b_3 \leftrightarrow (V_x, V_y, V_z)$. Similarly, angular velocity can be given in inertial frame by $\mathbf{\omega} = \omega_x e_1 + \omega_y e_2 + \omega_z e_3 \leftrightarrow (\omega_x, \omega_y, \omega_z)$, and in body-fixed frame by $\mathbf{\Omega} = \Omega_x b_1 + \Omega_y b_2 + \Omega_z b_3 \leftrightarrow (\Omega_x, \Omega_y, \Omega_z)$. The actual mass
and principle moment of inertia of the ellipsoid expressed in body frame are given by

\[
m_B^B = \frac{4}{3} \rho \pi abc,
\]

\[
J_B^B = m_B^B \begin{bmatrix}
\frac{(b^2 + c^2)}{5} & 0 & 0 \\
0 & \frac{(c^2 + a^2)}{5} & 0 \\
0 & 0 & \frac{(a^2 + b^2)}{5}
\end{bmatrix} \equiv \text{diag}(J_B^B_x, J_B^B_y, J_B^B_z).
\]

In general, the added mass $M_F^B$ must be solved numerically for an arbitrary shaped body. Fortunately, a single ellipsoid is one of the special cases whose potential $\phi$ (hence the added mass) can be analytically solved. Due to symmetry, one can readily show that $\Theta^{\phi x}$ and $\Theta^{x \phi}$ are both identically zero, while $\Theta^{\phi \phi}$ and $\Theta^{x x}$ are both diagonal, denote

\[
M_F^B \equiv \Theta^{\phi \phi} = \text{diag}(m_F^x, m_F^y, m_F^z),
\]

\[
J_F^B \equiv \Theta^{x x} = \text{diag}(J_F^x, J_F^y, J_F^z).
\]

Following Lamb [54], we define

\[
\alpha_0 = abc \int_0^\infty \frac{d\lambda}{(a^2 + \lambda)\Delta}, \quad \beta_0 = abc \int_0^\infty \frac{d\lambda}{(b^2 + \lambda)\Delta}, \quad \gamma_0 = abc \int_0^\infty \frac{d\lambda}{(c^2 + \lambda)\Delta},
\]

where

\[
\Delta = \sqrt{(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)}.
\]
Then, 

\[ m^F_x = \frac{\alpha_0}{2 - \alpha_0} m^B, \quad m^F_y = \frac{\beta_0}{2 - \beta_0} m^B, \quad m^F_z = \frac{\gamma_0}{2 - \gamma_0} m^B, \]

\[ J^F_x = \frac{m^B}{5} \frac{(b^2 - c^2)^2(\gamma_0 - \beta_0)}{2(b^2 - c^2) + (b^2 + c^2)(\beta_0 - \gamma_0)}, \]

\[ J^F_y = \frac{m^B}{5} \frac{(c^2 - a^2)^2(\alpha_0 - \gamma_0)}{2(c^2 - a^2) + (c^2 + a^2)(\gamma_0 - \alpha_0)}, \]

\[ J^F_z = \frac{m^B}{5} \frac{(a^2 - b^2)^2(\beta_0 - \alpha_0)}{2(a^2 - b^2) + (a^2 + b^2)(\alpha_0 - \beta_0)}. \]

We note that, for example,

\[ m^F_x - m^F_y = \frac{2(\alpha_0 - \beta_0)}{(2 - \alpha_0)(2 - \beta_0)} m^B, \]

therefore if \( a > b \), \( m^F_x < m^F_y \). One can readily show that

\[ a > b > c \quad \Rightarrow \quad m^F_x < m^F_y < m^F_z, \]

as might have been anticipated by the reader.

Finally, we re-emphasize that the total mass matrix is given in component form in the following expression

\[ M = M^B + M^F = \begin{bmatrix} m^B I_{3 \times 3} + M^F & 0 \\ 0 & J^B + J^F \end{bmatrix} \equiv \begin{bmatrix} M & 0 \\ 0 & J \end{bmatrix}, \quad (6.18) \]
where $M$ and $J$ are referred to as the total mass and total inertia of the solid-fluid system, respectively.

**Newtonian approach for the equations of motion** Just as one defines linear and angular momenta of a body in the space, one can also define linear and angular “momentum-like” quantities for the solid-fluid system, traditionally referred to as the *impulse* of the system (a term first introduced by Lord Kelvin). An impulse of the system is what required to counteract the impulsive pressures $\rho \phi$ exerted on the surface of the body and to generate the momenta of the body itself. Though the impulse is not equivalent to the total momenta of the system (which is indeterminate due to the infinite volume of the fluid), roughly speaking, it excludes the infinite term in the momenta, therefore it can be shown to vary as the momenta of a finite dynamical system would vary under the influence of external forces and torques, see Lamb [54] for a more rigorous discussion. With this understanding, in this work, we will simply refer to this impulse of system as “momentum”. In body fixed frame, linear momentum of the system $\mathbf{P}$ and angular momentum of the system $\mathbf{\Pi}$ are defined as the following

$$
\begin{pmatrix}
\mathbf{P} \\
\mathbf{\Pi}
\end{pmatrix} =
\begin{pmatrix}
\frac{\partial T}{\partial \mathbf{V}} \\
\frac{\partial T}{\partial \mathbf{\Omega}}
\end{pmatrix} =
\begin{pmatrix}
M \mathbf{V} \\
J \mathbf{\Omega}
\end{pmatrix}.
$$

(6.19)

Denote the corresponding linear momentum $\mathbf{p}$ and angular momentum $\mathbf{\pi}$ in inertial frame. While linear momentum is the total mass times linear velocity of mass center
of the body, angular momentum is not as straightforward: it must be defined with respect to certain reference point in the space, typically the origin of the reference frame. In (6.19), the body frame angular momentum is defined with respect to the mass center $C$. But in inertial frame, $\mathbf{\pi}$ is usually defined with respect to the origin $O$. The relation between the momenta expressed in these two frames is given by

$$ p = Q \mathbf{P} , \quad \mathbf{\pi} = Q \mathbf{\Pi} + x \times p . \tag{6.20} $$

We derive equations of motion for a single body submerged in perfect fluid using balance law of momenta. In absence of external force or torque, the governing equations in inertial frame are given by conservation of linear and angular momenta

$$ \dot{p} = 0 , \tag{6.21} $$

$$ \dot{\mathbf{\pi}} = 0 , \tag{6.22} $$

To obtain equations in the body-fixed frame, substitute (6.20) into the above equations. From (6.21),

$$ \dot{p} = 0 = \dot{Q} \mathbf{P} + Q \dot{\mathbf{P}} \quad \Rightarrow $$

$$ \dot{\mathbf{P}} = -Q^T \dot{Q} \mathbf{P} $$

$$ = -\hat{\Omega} \mathbf{P} = \mathbf{P} \times \Omega . $$
And from (6.22),
\[
\dot{\pi} = 0 = \dot{Q}\Pi + Q\dot{\Pi} + \dot{x} \times p + x \times \dot{p} \quad \Rightarrow \\
\dot{\Pi} = -Q^T \dot{Q}\Pi - Q^T (v \times p) \\
= -\hat{\Omega}\Pi - (Q^T v) \times (Q^T p) \\
= \Pi \times \Omega + P \times V .
\]

In summary, the equations of motion in body frame are given by
\[
\begin{align*}
\dot{P} &= P \times \Omega, \quad \text{(6.23)} \\
\dot{\Pi} &= \Pi \times \Omega + P \times V . \quad \text{(6.24)}
\end{align*}
\]

This set of equations is called the *Kirchhoff* equations.

After solving Kirchhoff equations, one obtains the velocities as functions of time. The position of the mass center and orientation of the body can then be integrated in time using the following equations
\[
\dot{Q} = Q \hat{\Omega}, \quad \dot{x} = Qv . \quad \text{(6.25)}
\]

This set of equation is called the *reconstruction* equations.
6.3 Equations of motion for the $N$-link fish model

Two approaches can be taken to model the submerged articulated fish. One is to assume the submerged links are *hydrodynamically decoupled*, as done in, for example, Radford [89]. This assumption means that the added mass associated with a given link is not affected by the presence of the other links. Added mass of each link is always in the form given by (6.17). And the links are only coupled by the mechanical constraints via the hinge joints. Clearly, this is unrealistic for a fish model. However, this approach is very useful for assessing qualitatively the behavior of the model, as the analytical solution for added mass is available. Another approach is to accurately compute the added mass of this articulated body so that the links are not only mechanically coupled, but also *hydrodynamically coupled*. We will present both approaches for the models in 2D, and only present the hydrodynamically decoupled approach in 3D.

6.3.1 Equations of motion for the $N$-link fish model in 3D

As mentioned before, we only show the hydrodynamically decoupled approach in 3D. First, we derive equations for the simplest case of multi-link model in 3D: a two-link fish. Then we generalize to the $N$-link model, especially the three-link model in 3D. We follow the Newtonian approach similar to the single ellipsoid case.

Two-link model in 3D As an example of the general $N$-link model, we show details of derivation for the two-link model in 3D depicted in Figure 6.3. The
Figure 6.3: Two-link model of fish in 3D, see text for details.

Orientation and position of $B_1$ (mass center $C_1$) and $B_2$ (mass center $C_2$) are given by $(Q_1, x_1)$ and $(Q_2, x_2)$, respectively. The position of hinge joint $K_1$ can be expressed in inertial frame $x_{K_1} = x_{K_1} e_1 + y_{K_1} e_2 + z_{K_1} e_3$, or equivalently $(x_{K_1}, y_{K_1}, z_{K_1})$, in the $B_1$-fixed frame

$$X_{K_1}^1 = -lb_1^1 + 0b_2^1 + 0b_3^1 \iff X_{K_1}^1 = (-l, 0, 0),$$

and in $B_2$-fixed frame

$$X_{K_1}^2 = lb_1^2 + 0b_2^2 + 0b_3^2 \iff X_{K_1}^2 = (l, 0, 0).$$

The shape of fish is parametrized by the relative angle $\theta_1$ between $b_1^2$ and $b_1^1$. The plane spanned by $b_1^1$ and $b_2^1$ coincides with the plane spanned by $b_1^2$ and $b_2^2$ due to the presence of the hinge joint. In another word, $b_3^1$ is parallel to $b_3^2$. The relative
orientation of $B_2$ with respect to $B_1$ is a rotation of angle $\theta_1$ about the direction $b_3^1$ (or equivalently $b_3^2$), which is given by

$$R_{21} = Q_1^T Q_2 = \begin{bmatrix} \cos \theta_1 & -\sin \theta_1 & 0 \\ \sin \theta_1 & \cos \theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$  \hfill (6.26)

Note that

$$R_{21}^T = R_{21}^{-1} = R_{12}.$$ 

Based on the hydrodynamical decoupling assumption, each link can be treated very similar to the case of a single ellipsoid. We will continue to think of the links as two identical “augmented” bodies whose total masses are given by (6.18). However, since $B_1$ and $B_2$ are coupled via the hinge joint $K_1$, the linear and angular momenta of each body are no longer always conserved, but the total momenta of the whole system are conserved. The two bodies exert equal and opposite forces and torques onto each other, see Figure 6.3. In inertial frame, denote the force and torque exerted from $B_2$ to $B_1$ as $f_{21}$ and $\tau_{21}$, respectively, and vice versa $f_{12}$ and $\tau_{12}$. One has

$$f_{21} = -f_{12}, \quad \tau_{21} = -\tau_{12}.$$ \hfill (6.27)
Note that the forces are due to the connection of the massless rods at the frictionless joint, and the torques are due to the torsional spring at the joint. The governing equations for the two links can be expressed in inertial frame by

\[
\begin{align*}
\dot{p}_1 &= f_{21} \\
\dot{\pi}_1 &= x_{K_1} \times f_{21} + \tau_{21} \\
\dot{p}_2 &= f_{12} \\
\dot{\pi}_2 &= x_{K_1} \times f_{12} + \tau_{12}
\end{align*}
\tag{6.28}
\]

These equations can also be expressed in their corresponding body frame. We denote the force and torque exerted on $B_1$ as $F_{21}$ and $T_{21}$ in its body frame, respectively. Similarly, $F_{12}$ and $T_{12}$ in $B_2$ frame. Since the torques are due to the spring, one has

\[
T_{21} = \begin{pmatrix} 0 \\ 0 \\ -k_1 \theta_1 \end{pmatrix}, \quad T_{12} = \begin{pmatrix} 0 \\ 0 \\ k_1 \theta_1 \end{pmatrix}.
\tag{6.29}
\]

The relation of equal and opposite forces and torques is now given in body frame variables

\[
F_{21} = -R_{21} F_{12}, \quad T_{21} = -R_{21} T_{12}.
\tag{6.30}
\]
Following similar procedure as in the single body case, the momenta equations can be given in their corresponding body frames

\[
\begin{align*}
\dot{P}_1 &= P_1 \times \Omega_1 + F_{21} \\
\dot{\Pi}_1 &= \Pi_1 \times \Omega_1 + P_1 \times V_1 + X_{K_1}^1 \times F_{21} + T_{21} \\
\dot{P}_2 &= P_2 \times \Omega_2 + F_{12} \\
\dot{\Pi}_2 &= \Pi_2 \times \Omega_2 + P_2 \times V_2 + X_{K_1}^2 \times F_{12} + T_{12}
\end{align*}
\] (6.31)

where \( P_i = MV_i, \Pi_i = J\Omega_i, i = 1,2 \), since the ellipsoids are identical, and

\[
M = \text{diag}(M_x, M_y, M_z), \quad J = \text{diag}(J_x, J_y, J_z)
\]

are the total mass and total moment of inertia matrices of a single ellipsoid, respectively. Due to the hydrodynamical decoupling assumption, the bodies are only mechanically constrained by the joint. Substitute (6.26) into (6.6) and (6.7), after some algebraic derivation, the relations between velocities are given by

\[
\Omega_2 = R_{12}\Omega_1 - \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad V_2 = R_{12}V_1 + (R_{12}\Omega_1) \times \begin{pmatrix} -l(1 + \cos \theta_1) \\ l\sin \theta_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ l\dot{\theta}_1 \\ 0 \end{pmatrix}
\] (6.32)

Therefore, the full sets of equations of motion are given by (6.29)-(6.32). Obviously, this set of equations need to be further simplified. Indeed, the 7 independent variables in the system are: linear and angular velocities (or equivalently, linear
and angular momenta) of one of the bodies, say, $V_1$ and $\Omega_1$ (or equivalently, $P_1$ and $\Pi_1$); and the relative angle $\theta_1$ as well as its derivative $\dot{\theta}_1$. The choice of using velocities of which body does not affect the solution of dynamics. Denote the independent variables compactly as $W = [V_{1x} V_{1y} V_{1z} \Omega_{1x} \Omega_{1y} \Omega_{1z} \theta_1 \dot{\theta}_1]^T$, after some straightforward but tedious simplifications, the governing equations (6.29)-(6.32) can be compactly written as

$$\dot{W} = g_2(W),$$

(6.33)

where $g_2$ is a vector valued nonlinear function, and the subscript $2$ is to indicate this function is for the two-link model in 3D. Note that one of the equation is trivial: $\dot{\theta}_1 = \dot{\theta}_1$, yet it is still convenient to keep both $\theta_1$ and $\dot{\theta}_1$ in the equations. After solving the equations for velocities and relative angle, one can then solve for the positions and orientations for the two bodies from the reconstruction equations for the two-link model

$$\dot{Q}_1 = Q_1 \Omega_1, \quad \dot{x}_1 = Q_1 V_1, \quad Q_2 = Q_1 R_{21}, \quad x_2 = x_1 - Q_2 X^2_{K_1} + Q_1 X^1_{K_1}. \quad (6.34)$$

**Generalization to the $N$-link model** For a $N$-link model in 3D, the positions of the joints on both sides of the $i^{th}$ body are expressed in its body-fixed frame as

$$X^i_{K_i} = (-l, 0, 0), \quad X^i_{K_{i+1}} = (l, 0, 0).$$
The relative orientation between any two bodies \( B_i \) and \( B_j \) can be given by \( R_{ij} \) which transforms from \( B_i \) to \( B_j \), i.e.

\[
R_{ij} = Q_j^T Q_i = \begin{bmatrix}
\cos \theta_{ij} & -\sin \theta_{ij} & 0 \\
\sin \theta_{ij} & \cos \theta_{ij} & 0 \\
0 & 0 & 1
\end{bmatrix}, \quad \theta_{ij} = \sum_{\alpha=j}^{i-1} \theta_{\alpha}.
\]

The constraint forces and torques exist in pair between the adjacent bodies, and they are equal and opposite similar to (6.27) and (6.30). Balance law of linear and angular momenta of the first and last bodies should have the same form as in (6.28), since the applied force and torque only come from one joint. But for \( B_i, 1 < i < N \), forces and torques are from both joints on the sides, therefore the balance law is given in the inertial frame as follows

\[
\begin{aligned}
\dot{p}_i &= f_{i-1,i} + f_{i+1,i} \\
\dot{\pi}_i &= \mathbf{x}_{Ki-1} \times f_{i-1,i} + \mathbf{x}_{Ki} \times f_{i+1,i} + \tau_{i-1,i} + \tau_{i+1,i}
\end{aligned}
\]

and the equations in the body frames should be altered accordingly. The relations between the forces and torques is given in body frame

\[
F_{i+1,i} = -R_{i+1,i} F_{i,i+1}, \quad T_{i+1,i} = -R_{i+1,i} T_{i,i+1}.
\]
And the torques are due to the torsional springs at the hinge joints, namely

\[
\begin{align*}
T_{i+1,i} &= \begin{pmatrix} 0 \\ 0 \\ -k_i \theta_i \end{pmatrix}, \\
T_{i,i+1} &= \begin{pmatrix} 0 \\ 0 \\ k_i \theta_i \end{pmatrix}.
\end{align*}
\]

For the \( N \)-link model, the independent variables are: linear and angular velocities (momenta) of one of the bodies in its corresponding body-fixed frame (6 components), and \( N - 1 \) relative angles as well as their time derivatives. Therefore, one has \( N + 5 \) independent variables in total. We emphasize once again that the choice of which body as a reference body is arbitrary and does not affect the dynamics. In fact, the independent velocities do not have to be associated with the mass center of one of the bodies. Without loss of generality, we choose \( B_1 \) as the reference body.
in $N$-link model. The relation between velocities of $B_i$ and the velocities of $B_1$ is
given by

\[
\begin{align*}
\Omega_i &= R_{1i} \Omega_1 - \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_{i1} \end{pmatrix}, \\
V_i &= R_{1i} V_1 + (R_{1i} \Omega_1) \times \left( -\sum_{\alpha=2}^{i} R_{\alpha i} X_{K_{\alpha-1}} + \sum_{\alpha=1}^{i-1} R_{\alpha i} X_{K_{\alpha}} \right) \\
&\quad + \sum_{\alpha=2}^{i} \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_{\alpha1} \end{pmatrix} \times \left( -R_{\alpha i} X_{K_{\alpha}} + R_{\alpha i} X_{K_{\alpha-1}} \right) + \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_{i1} \end{pmatrix} \times X_{K_{1i}}.
\end{align*}
\]

Therefore, we write the independent variables compactly as $W = [V_1^T \Omega_1^T \theta_1 \cdots \theta_{N-1} \dot{\theta}_1 \cdots \dot{\theta}_{N-1}]^T$, the governing equations can be written compactly as

\[
\dot{W} = g_N(W),
\]

where the vector valued nonlinear function $g_N$ has a subscript $N$ to indicate this is for the $N$-link model in 3D. After solving the equations for the velocities and relative angle, one can then solve for the positions and orientations of the $N$ bodies from the reconstruction equations

\[
\dot{Q}_1 = Q_1 \Omega_1, \quad \dot{x}_1 = Q_1 V_1, \quad Q_i = Q_1 R_{1i}, \quad x_i = x_1 + \sum_{j=1}^{i-1} \left( -Q_{j+1} X_{K_j}^{j+1} + Q_j X_{K_j}^j \right).
\]
Three-link model in 3D  We explicitly show the details of the equations for the three-link model in 3D. The governing equations are given in body-fixed frame variables by

\[
\begin{align*}
\dot{P}_1 &= P_1 \times \Omega_1 + F_{21} \\
\dot{\Pi}_1 &= \Pi_1 \times \Omega_1 + P_1 \times V_1 + X_{K_1}^1 \times F_{21} + T_{21} \\
\dot{P}_2 &= P_2 \times \Omega_2 + F_{12} + F_{32} \\
\dot{\Pi}_2 &= \Pi_2 \times \Omega_2 + P_2 \times V_2 + X_{K_1}^2 \times F_{12} + T_{12} + X_{K_2}^2 \times F_{32} + T_{32} \\
\dot{P}_3 &= P_3 \times \Omega_3 + F_{23} \\
\dot{\Pi}_3 &= \Pi_3 \times \Omega_3 + P_3 \times V_3 + X_{K_2}^3 \times F_{23} + T_{23}
\end{align*}
\]

The relations between forces and torques, relative orientation and the expression for torques are all in the same form as the general case. We choose $B_2$ as the reference body in the three-link model due to symmetry. The system can be reduced to in
terms of \( V_2 \) and \( \Omega_2 \), and the relative angles \( \theta_1 \) and \( \theta_2 \) as well as their derivative \( \dot{\theta}_1 \) and \( \dot{\theta}_2 \). The relations between velocities of \( B_1 \), \( B_3 \) and \( B_2 \) are given by

\[
\begin{align*}
\Omega_1 &= R_{21} \Omega_2 + \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{pmatrix}, \\
V_1 &= R_{21} V_2 + (R_{21} \Omega_2) \times \begin{pmatrix} -l(1 + \cos \theta_1) \\ l \sin \theta_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ l \dot{\theta}_1 \\ 0 \end{pmatrix} \\
\Omega_3 &= R_{23} \Omega_2 - \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{pmatrix}, \\
V_3 &= R_{23} V_2 + (R_{23} \Omega_2) \times \begin{pmatrix} l(1 + \cos \theta_2) \\ l \sin \theta_2 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ l \dot{\theta}_2 \\ 0 \end{pmatrix}
\end{align*}
\]

Therefore, we write the independent variables compactly as \( W = [V_2x \ V_2y \ V_2z \ \Omega_2x \ \Omega_2y \ \Omega_2z \ \theta_1 \ \theta_2 \ \dot{\theta}_1 \ \dot{\theta}_2]^T \), the governing equations can be written compactly as

\[
\dot{W} = g_3(W).
\] (6.36)

After solving the equations for velocities and relative angle, one can then solve for the positions and orientations for the 3 bodies from the reconstruction equations

\[
\begin{align*}
\dot{Q}_2 &= Q_{12} \dot{\Omega}_2, & \dot{x}_2 &= Q_2 V_2, \\
Q_1 &= Q_2 R_{12}, & x_1 &= x_2 + Q_2 X_{K_1}^2 - Q_1 X_{K_1}^1, \\
Q_3 &= Q_2 R_{32}, & x_3 &= x_2 - Q_3 X_{K_2}^3 + Q_2 X_{K_2}^2.
\end{align*}
\]
6.3.2 Equations of motion for the $N$-link fish model in 2D

So far, we have derived equations for the models in 3D, and they are generally quite complex. One can simplify the model further more by modeling the fish in 2D, therefore focus on the most important aspect of the system, as will be seen in the stability analysis. Similar to the 3D case, we model the fish in 2D as a neutrally buoyant articulated body moving in an infinitely large volume of perfect fluid $F$ at rest at infinity. Let the body be made of $N$ identical rigid ellipses (labeled $B_i$, $i = 1, \ldots, N$) of uniform density $\rho$ (equal to that of the fluid) with semi-axes $a$ and $b$. The ellipses are connected via $N - 1$ massless and frictionless hinge joints $K_i$ placed distance $l$ away from the centers of the ellipses along their major axes ($l > a$). The shape change of articulated body is parametrized by the relative angles $\theta_i$ between $B_i$ and $B_{i+1}$. Muscle elasticity of the fish is accounted for by torsional springs with stiffness constants $k_i$ between $B_i$ and $B_{i+1}$. An example of a two-link body in 2D is depicted in Figure 6.4.

![Figure 6.4: Two-link model of the fish in 2D, see text for details.](image-url)
Lagrangian approach for the two-link model in 2D  An inertial frame 
\{e_k\}_{k=1,2} is originated at \( O \), and a body-fixed frame \( \{b^i_k\}_{k=1,2} \) is attached to the 
center of each ellipse \( C_i \), where \( b^i_1 \) and \( b^i_2 \) are parallel to the \( a \) and \( b \) semi-axes of 
\( B_i \), respectively. The position of \( C_i \) in inertial frame is given by \( x_i = (x_i, y_i) \) and the 
orientation of \( B_i \) is parametrized by the angle \( \beta_i \) between \( e_1 \) and \( b^i_1 \), \( \theta_1 = \beta_1 - \beta_2 \). 
The transformation between the inertial frame and body frame is the following: 
if \( X = (X,Y) \) is any material point on \( B_i \) expressed in its body-fixed frame, and 
\( x = (x,y) \) is the same point expressed in inertial frame, the relation is given by 
\[
    x = Q_i X + x_i, \quad Q_i = \begin{bmatrix}
        \cos \beta_i & -\sin \beta_i \\
        \sin \beta_i & \cos \beta_i
    \end{bmatrix}
\]
Linear velocity of \( C_i \) in inertial frame and body frame are given by 
\[
    v_i = (v_{ix}, v_{iy}) = (\dot{x}_i, \dot{y}_i), \quad V_i = (V_{ix}, V_{iy}),
\]
and the transformation between them is given by
\[
    v_i = Q_i V_i.
\]
Angular velocity of \( B_i \) is denoted as \( \omega_i \) in inertial frame, while in body frame it is 
denoted as \( \Omega_i \). Note that the direction of angular velocity is indeed perpendicular
to the plane of motion given by the right hand rule. Since \( \{e_k\} \) and \( \{b^i_k\} \) span the same plane, angular velocity in both frames are in fact identical, i.e.

\[
\omega_i \equiv \Omega = \dot{\beta}_i.
\]

We now derive the constraint equation due to the hinge joint. The position of the joint \( K_1 \) is given by \( X_{K_1}^1 = (-l, 0) \) in the \( B_1 \)-fixed frame and \( X_{K_1}^2 = (l, 0) \) in the \( B_2 \)-fixed frame. The constraint is given by

\[
Q_1X_{K_1}^1 + x_1 = Q_2X_{K_1}^2 + x_2,
\]

take time derivative of above equation,

\[
\dot{Q}_1X_{K_1}^1 + Q_1\dot{X}_{K_1}^1 + \dot{x}_1 = \dot{Q}_2X_{K_1}^2 + Q_2\dot{X}_{K_1}^2 + \dot{x}_2 \Rightarrow
\]

\[
\omega_1 \frac{\partial Q_1}{\partial \beta_i} X_{K_1}^1 + v_1 = \omega_2 \frac{\partial Q_2}{\partial \beta_i} X_{K_1}^2 + v_2,
\]

Denote

\[
S_i = \frac{\partial Q_i}{\partial \beta_i} = \begin{bmatrix}
-\sin \beta_i & -\cos \beta_i \\
\cos \beta_i & -\sin \beta_i
\end{bmatrix},
\]

the constraint equation in terms of the inertial frame velocities is given by

\[
\omega_1 S_1 X_{K_1}^1 + v_1 = \omega_2 S_2 X_{K_1}^2 + v_2,
\]

(6.37)
In component form, (6.37) is given by

\[
\begin{pmatrix}
v_{2x} \\
v_{2y}
\end{pmatrix} = \begin{bmatrix}
v_{1x} + (\dot{\beta}_1 - \dot{\theta}_1)l\sin(\beta_1 - \theta_1) + \dot{\beta}_1l\sin\beta_1 \\
v_{1y} - (\dot{\beta}_1 - \dot{\theta}_1)l\cos(\beta_1 - \theta_1) - \dot{\beta}_1l\cos\beta_1
\end{bmatrix}.
\]

Since we can now easily express the orientation of each body by \( \beta_i \), it is more convenient to use the Lagrangian approach in order to derive equations of motion. More importantly, this derivation process also paves the way for the hydrodynamically coupled case. The Lagrangian function \( L \) is equal to the total kinetic energy minus the potential energy of the system, namely, \( L = T - U \). Since the articulated body is neutrally buoyant, potential energy is only due to the presence of spring, i.e.

\[
U = \frac{1}{2}k_1\theta_1^2. \tag{6.38}
\]

The kinetic energy \( T \) of the system can be written as the sum of the energies of the ellipses \( T_{Bi}, i = 1, 2 \) and the energy of the fluid \( T_F \), namely \( T = \sum_{i=1}^{2} T_{Bi} + T_F \). The body kinetic energy is given in terms of the velocities in inertial frame

\[
T_{Bi} = \frac{1}{2} \sum_{i=1}^{2} \left( v_i^T m^B v_i + \omega_i J^B \omega_i \right), \tag{6.39}
\]

where

\[
m_i^B = \rho \pi ab, \quad J^B = \frac{a^2 + b^2}{4} m_i^B
\]
are the actual mass and moment of inertia of an ellipse. Since the model is assumed to be hydrodynamically decoupled, the added mass around each body should have the same form of a single isolated body. In 3D, added mass of a single ellipsoid is given by (6.17), and one should expect similar form of added mass in 2D, namely, $M^F = \text{diag}(m_x^F, m_y^F)$, and $J^F$ is a scalar now. To be consistent with the body energy, kinetic energy of fluid can also be written in terms of velocities in inertial frame by first writing in body frame velocities then transforming into inertial frame velocities, i.e.

\[ T_F = \frac{1}{2} \sum_{i=1}^{2} (\mathbf{v}_i \cdot M^F \mathbf{v}_i + \Omega_i \cdot J^F \Omega_i) \]

\[ = \frac{1}{2} \sum_{i=1}^{2} \left( \mathbf{v}_i^T Q_i^T M^F Q_i \mathbf{v}_i + \omega_i J^F \omega_i \right), \]

where the added mass $M^F$ and $J^F$ can be analytically solved for the ellipses (see, for example, Newman [82]),

\[ M^F = \begin{bmatrix} m_x^F & 0 \\ 0 & m_y^F \end{bmatrix} = \begin{bmatrix} \rho \pi b^2 & 0 \\ 0 & \rho \pi a^2 \end{bmatrix}, \quad J^F = \frac{1}{8} \rho \pi (a^2 - b^2)^2. \]

Therefore, denote the velocities of $B_i$ in inertial frame as $\mathbf{w}_i = [v_{ix} \ v_{iy} \ \omega_i]^T$, Lagrangian function can be compactly written as

\[ L = \frac{1}{2} \sum_{i=1}^{2} \mathbf{w}_i^T \tilde{M} \mathbf{w}_i - \frac{1}{2} k_1 \theta_i^2, \quad \tilde{M} = \begin{bmatrix} m^B I_{2 \times 2} + Q_i^T M^F Q_i & 0 \\ 0 & J^B + J^F \end{bmatrix}. \quad (6.41) \]
The inertial frame velocities $\mathbf{w}_i$ and relative angles $\theta_1$ are not all independent. As mentioned before, the independent variables for the two-link model in 2D are: position and orientation of one of the bodies, say, $\mathcal{B}_1$, that is $(x_1, y_1)$ and $\beta_1$, and the relative angle $\theta_1$. Denote the configuration variables in inertial frame as $\mathbf{q} = [x_1 \ y_1 \ \beta_1 \ \theta_1]^T$, and the generalized velocity is $\dot{\mathbf{q}} = [\dot{x}_1 \ \dot{y}_1 \ \dot{\beta}_1 \ \dot{\theta}_1]^T$. To rewrite $L$ in terms of $\mathbf{q}$ and $\dot{\mathbf{q}}$, the inertial frame velocities $\mathbf{w}_i$ must be expressed in terms of $\dot{\mathbf{q}}$, and the transformation is given by (6.37). Explicitly, one has the following

$$\mathbf{w}_1 = B_1 \dot{\mathbf{q}}, \quad B_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$\mathbf{w}_2 = B_2 \dot{\mathbf{q}}, \quad B_2 = \begin{bmatrix} 1 & 0 & l \sin \theta_1 + l \sin(\beta_1 - \theta_1) & l \sin(\beta_1 - \theta_1) \\ 0 & 1 & -l \cos \theta_1 - l \cos(\beta_1 - \theta_1) & -l \cos(\beta_1 - \theta_1) \\ 0 & 0 & 1 & 1 \end{bmatrix} \cdot$$

(6.42)

Substitute above equations into the Lagrangian function, one gets the Lagrangian function in terms of $\mathbf{q}$ and $\dot{\mathbf{q}}$

$$L = \frac{1}{2} \dot{\mathbf{q}}^T \left( \sum_{i=1}^{2} B_i^T \tilde{\mathbf{M}}_i B_i \right) \dot{\mathbf{q}} - \frac{1}{2} k_1 \theta_1^2 \equiv \frac{1}{2} \dot{\mathbf{q}}^T \mathbf{M}_{\text{total}} \dot{\mathbf{q}} - \frac{1}{2} \dot{\mathbf{q}}^T K \mathbf{q},$$

(6.43)

where $\mathbf{M}_{\text{total}} = \sum_{i=1}^{2} B_i^T \tilde{\mathbf{M}}_i B_i$ depends only on $\beta_1$ and $\theta_1$, and $K = \text{diag}(0, 0, 0, k_1)$. 
Since there is no external force or torque applied to the body-fluid system, equations of motion can be given by the Euler-Lagrange equations,

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0. \tag{6.44}
\]

Substitute the Lagrangian function into (6.44), one gets the equations of motion

\[
\ddot{q} + \dot{M}_{\text{total}} \dot{q} - \frac{1}{2} \dot{q}^T \frac{\partial M_{\text{total}}}{\partial q} \dot{q} + Kq = 0. \tag{6.45}
\]

Here the time and spatial derivatives of \(M_{\text{total}}\) should be passed into \(B_i\) via the chain rule. This is a set of 4 second-order differential equations.

**Generalization to the \(N\)-link model in 2D** For the \(N\)-link model in 2D, we follow the similar procedure as the two-link model. Here we only point out the key steps that are distinguished from before. The Lagrangian function \(L\) is given in inertial frame variables as

\[
L = \frac{1}{2} \sum_{i=1}^{N} w_i \cdot \tilde{M}_i w_i - \frac{1}{2} \sum_{i=1}^{N-1} k_i \theta_i^2.
\]
The constraint equation in terms of the inertial velocities can be given by a general form which uses the velocity of $B_1$ as a reference velocity

$$v_i = v_1 + \sum_{\alpha=1}^{i-1} (-\omega_{\alpha+1} S_{\alpha+1} X_{K_{\alpha}} + \omega_{\alpha} S_{\alpha} X_{K_{\alpha}})$$

$$= \begin{pmatrix} v_{1x} \\ v_{1y} \end{pmatrix} + l \sum_{\alpha=1}^{i-1} \left[ \omega_{\alpha+1} \begin{pmatrix} \sin \beta_{\alpha+1} \\ -\cos \beta_{\alpha+1} \end{pmatrix} + \omega_{\alpha} \begin{pmatrix} \sin \beta_{\alpha} \\ -\cos \beta_{\alpha} \end{pmatrix} \right]$$

$$= \begin{pmatrix} v_{1x} + l \dot{\beta}_1 \sin \beta_1 + l \dot{\beta}_i \sin \beta_i + 2l \sum_{\alpha=2}^{i-1} \dot{\beta}_\alpha \sin \beta_\alpha \\ v_{1y} - l \dot{\beta}_1 \cos \beta_1 - l \dot{\beta}_i \cos \beta_i - 2l \sum_{\alpha=2}^{i-1} \dot{\beta}_\alpha \cos \beta_\alpha \end{pmatrix},$$

where

$$\beta_i = \beta_1 - \sum_{\alpha=1}^{i-1} \theta_i.$$
where $B_i$ takes the form

$$
\begin{bmatrix}
1 & 0 & l(s_1 + s_i + 2 \sum_{a=2}^{i-1} s_a) & -l(s_i + 2 \sum_{a=2}^{i-1} s_a) & -l(s_i + 2 \sum_{a=2}^{i-1} s_a) \\
\vdots & -l s_i & 0 & \vdots & 0 \\
0 & 1 & -l(c_1 + c_i + 2 \sum_{a=2}^{i-1} c_a) & l(c_i + 2 \sum_{a=2}^{i-1} c_a) & l(c_i + 2 \sum_{a=2}^{i-1} c_a) \\
\vdots & l c_i & 0 & \vdots & 0 \\
0 & 0 & 1 & -1 & -1 \\
\vdots & -1 & 0 & \vdots & 0
\end{bmatrix}
$$

and

$$s_i = \sin \beta_i, \quad c_i = \cos \beta_i.$$

And the Lagrangian function in inertial frame is given by

$$L = \frac{1}{2} \dot{q}^T \left( \sum_{i=1}^{N} B_i^T \tilde{M} B_i \right) \dot{q} - \frac{1}{2} \sum_{i=1}^{N-1} k_i \theta_i^2 \equiv \frac{1}{2} \dot{q}^T \bar{M}_{\text{total}} \dot{q} - \frac{1}{2} \dot{q}^T K \dot{q}, \quad (6.46)$$

where $\bar{M}_{\text{total}}$ depends on $\beta_1$ and $\theta_i$, and $K = \text{diag}(0, 0, 0, k_1, \ldots, k_{N-1})$. The governing equation is given by the same form as in (6.45).

**Equations of motion for the $N$-link hydrodynamically coupled model in 2D** Now the ellipses are assumed to be not only mechanically coupled, but also
hydrodynamically coupled. The general procedure is very similar to the hydrodynamically decoupled case, and we only point out the different steps. The potential energy is now

\[ U = \frac{1}{2} \sum_{i=1}^{N-1} k_i \theta_i^2. \]

And the kinetic energy of the fish body is similar to the decoupled case, which is given by

\[
T_B = \frac{1}{2} \dot{\mathbf{q}} \cdot \sum_{i=1}^{N} B_i^T \begin{bmatrix} m^B & 0 & 0 \\ 0 & m^B & 0 \\ 0 & 0 & J^B \end{bmatrix} B_i \dot{\mathbf{q}} \equiv \frac{1}{2} \dot{\mathbf{q}} \cdot M^B \dot{\mathbf{q}}.
\]

The kinetic energy of the fluid is given by

\[
T_F = -\frac{1}{2} \rho \int_{\sum_{i=1}^{N} \partial B_i} \phi \nabla \phi \cdot \mathbf{n} \, ds,
\]

where the integration is over the boundary the the bodies \( \partial B_i \), \( \mathbf{n} \) is an outward (into the fluid) normal vector to \( \partial B_i \), and \( ds \) is an infinitesimal line element of \( \partial B_i \).

The potential \( \phi \) may be expressed in Kirchhoff form

\[
\phi = \dot{x}_1 \varphi + \dot{y}_1 \psi + \dot{\beta}_1 \chi_0 + \sum_{\alpha=1}^{i-1} \dot{\theta}_\alpha \chi_\alpha,
\]

where \( \varphi \) and \( \psi \) represent the translational velocity potentials in the \( \mathbf{e}_1 \) and \( \mathbf{e}_2 \) directions in inertial frame, respectively, and \( \chi_\alpha \) are the rotational velocity potentials.
Due to linearity, velocity potentials are also solutions of Laplace’s equation, and the
boundary conditions are given by zero gradient at infinity, and on each boundary
$\partial B_i$ satisfying the following conditions

\[
\frac{\partial \varphi}{\partial n} = \begin{pmatrix} e_1 \cdot n \\ \vdots \\ e_1 \cdot n \end{pmatrix}, \quad \frac{\partial \psi}{\partial n} = \begin{pmatrix} e_2 \cdot n \\ \vdots \\ e_2 \cdot n \end{pmatrix},
\]

\[
\frac{\partial \chi_0}{\partial n} = \begin{pmatrix} |(x_{B_1} - x_1) \times n| \\ \vdots \\ |(x_{B_N} - x_N) \times n| \end{pmatrix}, \quad \frac{\partial \chi_i}{\partial n} = \begin{pmatrix} 0 \\ \vdots \\ |(x_{B_{i+1}} - x_{i+1}) \times n| \\ \vdots \\ |(x_{B_N} - x_N) \times n| \end{pmatrix},
\]

(6.49)

where $x_i$ is the position of the $i^{th}$ body center in inertial frame, $x_{B_i}$ is the position of
an arbitrary material point on $\partial B_i$ expressed in inertial frame. Hence, the velocity
potentials depend only on the geometry of each body and the shape of the whole
articulated body. Substitute (6.48) into (6.47), one can readily verify that the
The kinetic energy of the fluid can be expressed in terms of the configurations and velocities of $B_i$ in the following form

\[
T_F = \frac{1}{2} \mathbf{q}^T \mathbf{M}_F \mathbf{q},
\]

where $\mathbf{M}_F$ is the added mass matrix. The entries of $\mathbf{M}_F$ are integrations of the velocity potentials over the boundaries. For example,

\[
\Theta^{\phi\psi} = \frac{1}{2} \rho \int \sum_{i=1}^{N} \partial B_i \left( \phi \frac{\partial \psi}{\partial n} + \frac{\partial \phi}{\partial n} \psi \right) ds.
\]

Note the added mass now integrates over the boundaries of all bodies, rather than over a single boundary as in the hydrodynamically decoupled case. Hence, the added mass now can only be associated with the articulated body as a whole, not with any ellipse individually. One can see that the added mass now depends on the relative angles $\theta_i$, i.e. $\mathbf{M}_F$ can be a function of time. This is precisely the reason why the ellipses are now hydrodynamically coupled.
Therefore, the Lagrangian function of the body-fluid system is given by

\[ L = T_B + T_F - U = \frac{1}{2} \dot{\mathbf{q}} \cdot (\mathbf{M}_B + \mathbf{M}_F) \dot{\mathbf{q}} - \frac{1}{2} \sum_{i=1}^{N-1} k_i \theta_i^2 \]

\[ \equiv \frac{1}{2} \dot{\mathbf{q}}^T \mathbf{M}_{\text{total}} \dot{\mathbf{q}} - \frac{1}{2} \mathbf{q}^T \mathbf{K} \mathbf{q}, \]  

(6.51)

where the total mass is given by \( \mathbf{M}_{\text{total}} = \mathbf{M}_F + \mathbf{M}_B \), which depends on \( \beta_1 \) and \( \theta_i \), and \( \mathbf{K} = \text{diag}(0, 0, k_1, \cdots, k_{N-1}) \). The total mass depends on \( \beta_1 \) only because we are expressing the problem in inertial frame. If one expresses the problem in the body-fixed frame variables, the total mass does not depend on the orientation of the whole body, but still depends on the relative angles, which indicates the shape of the fish model. Euler-Lagrange equation is also in the same form as before, which we repeat here

\[ \mathbf{M}_{\text{total}} \ddot{\mathbf{q}} + \dot{\mathbf{M}}_{\text{total}} \dot{\mathbf{q}} - \frac{1}{2} \dot{\mathbf{q}}^T \frac{\partial \mathbf{M}_{\text{total}}}{\partial \mathbf{q}} \dot{\mathbf{q}} + \mathbf{K} \mathbf{q} = 0. \]

This is a set of \( N + 2 \) second-order differential equations. The difference between decoupled and coupled cases can be demonstrated by the streamline plot of three-link models in both cases, see Figure 6.5.
Figure 6.5: Schematics of the streamlines in both hydrodynamically decoupled and coupled models. (Courtesy of E. Kanso.)
Chapter 7

2D Stability Analysis

The main focus of this work is to study the stability of the coast motion of fish. Hence we are interested in a family of relative equilibrium solutions of (6.45) for the $N$-link model: \( \dot{q}_e \equiv \begin{bmatrix} v_1 x_e & v_1 y_e & \dot{\beta}_1 & \dot{\theta}_1 & \cdots & \dot{\theta}_{N-1} \end{bmatrix}^T = \begin{bmatrix} v_e & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}^T, \forall v_e \in \mathbb{R}. \) This relative equilibrium corresponds to the motion that the $N$ rigid links translating along their major axis of symmetry at an arbitrary speed $v_e$, with relative angles always remaining zero and no rotation as well. This is precisely the coast motion of a $N$-link fish model in 2D. To analyze the stability of this relative equilibrium, we first use linearization of the system to obtain linear stability/instability based on certain parameters, then verify the findings using numerical simulations of the nonlinear equations (6.45) for initial conditions slightly perturbed from $\dot{q}_e$. If all eigenvalues of the linearized system have non-positive real parts, the corresponding relative equilibrium is said to be \textit{linearly} or \textit{marginally stable} (given that the system is conservative, i.e., no energy dissipation); if at least one eigenvalue has positive real part, the equilibrium is said to be linearly unstable. Note that one can conclude
nonlinear instability in Lyapunov’s sense from linear instability. But linear stability
does not necessarily indicate nonlinear stability, see, e.g., [67]. We present numerical
evidence that the system might indeed be nonlinearly stable where it is linearly
stable.

Before proceeding forward, we shall nondimensionalize the equations using length
scale \( a \), mass scale \( \rho \pi a^2 \), and time scale \( a/v_e \). All terms can be readily nondimen-
sionalized accordingly, for instance, one gets

\[
\begin{align*}
\tilde{a} &= 1, \quad \tilde{b} = \frac{b}{a}, \quad \tilde{l} = \frac{l}{a}, \quad \tilde{x}_i = \frac{x_i}{a}, \quad \tilde{v}_i = \frac{v_i}{v_e}, \quad \tilde{v}_e = 1, \\
\tilde{m}^B &= \frac{m^B}{\rho \pi a^2}, \quad \tilde{J}^B = \frac{J^B}{\rho \pi a^4}, \quad \tilde{k}_i = \frac{k_i}{\rho \pi a^2 v_e^2},
\end{align*}
\]

the entries in added mass can be nondimensionalized similarly as the actual mass
and inertia, and the angles are already in dimensionless form. For simplicity, we
drop the \( \tilde{() \)} notation with the understanding that all variables in the 2D case are
dimensionless hereafter.

### 7.1 2D stability analysis for the hydrodynamically decoupled model

We consider the hydrodynamically decoupled case first. The governing equations
are given by (6.45), which is a set of \( N + 2 \) second-order differential equations
for the \( N \)-link model. Traditionally, these equations can be rewritten as \( 2N + 4 \)
first-order differential equations in terms of the state variable

\[
\mathbf{\eta} = [\mathbf{q}^T \: \dot{\mathbf{q}}^T]^T =
\]
The Euler-Lagrange equations now take the form

\[
\begin{bmatrix}
\dot{\eta} \\
\ddot{\eta}
\end{bmatrix} = \begin{bmatrix}
\dot{\eta} \\
\ddot{\eta}
\end{bmatrix} = \begin{bmatrix}
\dddot{q} \\
\dddot{q}
\end{bmatrix} = \begin{bmatrix}
M_{total}^{-1} \left( -\dddot{M}_{total} q + \frac{1}{2} q^T \partial \dddot{M}_{total} \frac{\partial q}{\partial q} \right) \\
-K q
\end{bmatrix}
\]

The relative equilibrium is now \( \eta_e = [t \ 0 \ 0 \ \cdots \ 0 \ 1 \ 0 \ 0 \ \cdots \ 0]^T \), given the dimensionless coast speed is \( v_e = 1 \). To linearize (7.1) around \( \eta_e \), we write \( \eta = \eta_e + \delta \eta \), where \( \delta \) indicates an infinitesimal perturbation. The linearized equations of motion can be expressed as

\[
\delta \ddot{\eta} = A \delta \eta ,
\]

where \( A \) is called the Jacobian matrix or simply Jacobian, it is a \((2N+4) \times (2N+4)\) matrix. Among the rows of \( A \), \( N + 2 \) rows are in the form that the entries of each row should be all zeros except only one entry being 1, since these rows correspond to the fact \( \delta \dddot{q} = \delta \dddot{q} \) from both left and right hand sides of the equations.

To obtain the linearized equations of motion around \( \eta_e \) and evaluate the Jacobian, one may take two approaches: one approach is to substitute \( \eta = \eta_e + \delta \eta \) into (7.1), and eliminate higher order terms \( O(\|\delta \eta\|^2) \) from the equations, leaving only the linear terms. This analytical approach is certainly feasible, but the derivation can be tedious even for the simplest two-link 2D hydrodynamically decoupled model. For more complex cases, as can be imagined, analytical procedure is practically not operable. The other approach is to numerically evaluate the Jacobian.
Denote the governing equations (7.1) compactly as \( \dot{\eta} = f(\eta) \), the Jacobian is given by

\[
A = \frac{\partial f}{\partial \eta}(\eta_e),
\]

where \( f(\eta_e) = 0 \) by definition. One can numerically evaluate the entries of Jacobian up to any order of accuracy. Here we show a mid-point rule (which is a second order scheme) as an example: each column of \( A \) can be approximated by

\[
A(:,k) \approx \frac{f(\eta_e + \delta \eta_k) - f(\eta_e - \delta \eta_k)}{2\|\delta \eta_k\|}, \quad k = 1, \cdots, 2N + 4,
\]

where \( \delta \eta_k = [0 \cdots 0 \epsilon_k 0 \cdots 0]^T \), i.e. everywhere zero except only the \( k^{\text{th}} \) entry, which is an infinitesimal value \( \epsilon_k \). For simplicity, assume \( \epsilon_k \equiv \epsilon \) for all \( k \).

For concreteness, we will discuss the two-link and three-link cases in the following paragraphs as examples of the general \( N \)-link model.

**Two-link model** For the two-link model in 2D, the relative equilibrium is \( \eta_e = [t 0 0 0 1 0 0 0]^T \). The total mass in inertial frame \( M_{\text{total}} \) is a function of \( \beta_1 \) and \( \theta_1 \) only. Linearized equations can be given by (7.2), where \( A \) is now a \( 8 \times 8 \) matrix, which has 8 eigenvalues \( \lambda_i, i = 1, \cdots, 8 \). The eigenvalues can only be either
pure imaginary complex conjugates or equal and opposite real pairs, given the system is Hamiltonian. Two pairs of the eigenvalues are always 0, which reflects an $SE(2)$ symmetry in the equations (in the directions of $x_1$, $y_1$, $\beta_1$), and also the fact that $\eta_e$ is a relative equilibrium for any coast speed $v_e$ (in the direction of $\dot{x}_1$). The other eigenvalues depend on 3 independent parameters in the system: dimensionless aspect ratio $b$ of the two identical ellipses, spring stiffness $k_1$ and the distance from the hinge joint to mass center $l$ ($l > 1$). Since the articulated body is a model of fish, one would like to let $l \to 1$ as much as possible for the model to mimic a fish shape. In another word, one would like the gap between bodies to be as small as possible. However, if $l$ is too close to 1, the articulated body is limited to be able to only undergo a very small shape change before the boundaries of bodies touch. It turns out that $l = 1.1$ is an optimal parameter both physically and numerically, i.e. the length of rigid bar is one tenth of the semi-major axis $a = 1$. We will not vary the value of $l$ in this work, and set it to be $l = 1.1$ for all cases.

For a given $b$, e.g. $b = 0.2$, real parts of eigenvalues $\lambda_{real}$ are depicted in Figure 7.2(a). One identifies that $\lambda_{real}$ are all zeros for $b = 0.2$ and $0.257 \leq k_1 \leq 0.480$, which suggests linear stability for this parameter region. Follow the same procedure, explore the $(b, k_1)$ parameter space, the regions in which the system are linearly stable as shown in Figure 7.2(b). The case $b = 0.2$ is highlighted, in which the system is stable when $0.257 \leq k_1 \leq 0.480$, as depicted in Figure 7.2(a). Two linearly stable regions in the $(b, k_1)$ parameter space can be readily identified: one is the half plane $b \geq 1$, the other is the shark fin shaped area in $b < 1$. The half plane can be
Figure 7.2: Two-link hydrodynamically decoupled model in 2D: (a) Real part of eigenvalues $\lambda_{\text{real}}$ as functions of $k_1$ for $b = 0.2$. For $0.257 \leq k_1 \leq 0.480$, $\lambda_{\text{real}}$ are all zero. (b) Stable regions in $b - k_1$ parameter space. Two stable regions are found: one is the half plane $b \geq 1$, the other is the shark fin shaped area in $b < 1$. The case $b = 0.2$ is highlighted, the system is stable when $0.257 \leq k_1 \leq 0.480$.

Understand easily since a single ellipse is stable when $b > 1$, and the hydrodynamically decoupled two-link model is essentially a superposition of two ellipses albeit a mechanical constraint, which will not de-stabilize the system in this model. The second region is in contrast with the stability result of a single submerged body, which is unstable for all $b < 1$. This means, by allowing an internal shape change via $\theta_1$, the two-link model shows that linear stability can be achieved for $b < 1$ given an appropriate elasticity. Indeed, for a given $b$ in the stable region, if $k_1$ is small (the spring is weak), the model is in some sense similar to two independent ellipses, hence the system is unstable. If, on the other hand, $k_1$ is too large (the spring is stiff) and the model is similar to one elongated body, which is also unstable. For an intermediate range of $k_1$ values which provide the appropriate bending resistance to perturbations, the system is stable.
Figure 7.3: Numerical integration for the two-link hydrodynamically decoupled model. Initial condition is $\eta_0 = [t \ 0 \ 0.02 \ 1 \ 0 \ 0 \ 0]^T$. Integration is for time from 0 to 30, with parameter values $b = 0.2$ for all cases and $k_1 = 0.1$ (top row), 0.4 (middle row) and 0.7 (bottom row). The left plots show trajectories of the hinge joint in inertial frame in dashed line ($x$ scale is $[-4.5 \ 30.5]$), with snapshots of the fish at time $t = 0$, 18 and 30 overlaid. The right plots show orientation angle $\beta_1$ as functions of time. Only the $k_1 = 0.4$ case (middle row) is stable.

The linear stability results can be verified by direct numerical integration of the nonlinear equations with initial conditions slightly perturbed from $\eta_0$. For instance, integrate (7.1) from $t = 0$ to 30 with initial condition $\eta_0 = [t \ 0 \ 0.02 \ 1 \ 0 \ 0 \ 0]^T$, i.e. $\theta_1$ slightly perturbed from 0, for parameter values $b = 0.2$ and $k_1 = 0.1$, 0.4 and 0.7. The three distinct values of $k_1$ correspond to values for which the system is unstable ($k_1 = 0.1$), stable (0.4), and unstable (0.7), as predicted by the linear stability results in Figure 7.2. The results are depicted in Figure 7.3, where trajectories of the hinge joint in inertial frame and orientation $\beta_1$ as functions of time are shown. Snapshots of the body motion in inertial frame are also overlaid on top of the trajectories at $t = 0, 18$ and 30 for all three cases. Clearly, the equilibrium is stable only when $k_1 = 0.4$, while for $k_1 = 0.7$ the body tumbles and for $k_1 = 0.1$ the body tumbles and oscillates. One can explore all possible initial conditions and the same conclusion holds for all kinds of perturbations.
Figure 7.4: Relative equilibrium of the three-link model in 2D: fish with semi-axes 1 and $b$ translating along its major axis of symmetry with speed 1, no rotation, and relative angle is always zero.

**Three-link model** For the three-link model in 2D, let $\theta_1$ and $\theta_2$ be the relative angles between $B_1$, $B_2$ and $B_2$, $B_3$, respectively. The spring stiffnesses at the joints are $k_1$ and $k_2$. The relative equilibrium is now $\eta_e = [x_1 e \ y_1 e \ \beta_{1e} \ \theta_{1e} \ \theta_{2e} \ \dot{x}_{1e} \ \dot{y}_{1e} \ \dot{\beta}_{1e} \ \dot{\theta}_{1e} \ \dot{\theta}_{2e}]^T = [t \ 0 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0]^T$. The total mass in inertial frame $M_{total}$ is now a function of $\beta_1$, $\theta_1$ and $\theta_2$. Linearized equations are given by (7.2), where $A$ is a $10 \times 10$ matrix, and it has 10 eigenvalues $\lambda_i$, $i = 1, \cdots, 10$, which should be 5 complex conjugate (for pure imaginary) or equal and opposite signed (for pure real) pairs. Similar to the two-link case, two pairs of the eigenvalues are always 0. The other eigenvalues depend on 3 parameters $b$, $k_1$ and $k_2$. We shall follow the same procedure to obtain stable regions in the parameter space, which is now $(b, k_1, k_2)$.

As an example, we explore a cross-section $\mathcal{I}$ of the full $(b, k_1, k_2)$ space, that is when $k_1 = k_2$.

Figure 7.5: Stable regions in the $(b, (k_1 = k_2))$ plane for the three-link hydrodynamically decoupled model in 2D. Two stable regions are found: one is the half plane $b \geq 1$, the other is the shark fin shaped area in $b < 1$. 

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\( k_1 = k_2 \). The linearly stable and unstable regions on the \((b,(k_1 = k_2))\) plane are shown in Figure 7.5. Similar to the two-link case, two regions can be identified as linearly stable: one is the half plane \( b \geq 1 \), and the other is the shark fin shaped area in \( b < 1 \). These regions can be understood in the same sense as the two-link case. The shark fin area is \textit{larger} in comparison to the two-link case. Again, one could verify these linear stability results by direct numerical integration of the nonlinear equations with initial condition slightly perturbed from \( \eta_0 \). For instance, integrate (7.1) from \( t = 0 \) to 40 with initial condition \( \eta_0 = [t \ 0 \ 0.01 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0]^T \), i.e. \( \beta_1 \) slightly perturbed from 0, for parameter values \( b = 0.2 \) and \( k_1 = k_2 = 0.4, 0.8 \) and 1.2. The three distinct values of \( k_1 \) (and \( k_2 \)) correspond to values for which the system is unstable \((k_1 = k_2 = 0.4)\), stable \((0.8)\), and unstable \((1.2)\), as predicted by Figure 7.5. The results are depicted in Figure 7.6, where the trajectories of the center \( C_2 \) of \( B_2 \) in inertial frame and orientation \( \beta_2 = \beta_1 - \theta_1 \) as functions of time are shown. Snapshots of the body motion in inertial frame are also overlaid on top of the trajectories at \( t = 0, 20 \) and 40 for all three cases. Clearly, the equilibrium is stable only when \( k_1 = k_2 = 0.8 \), while for 1.2 the body tumbles and for 0.4 the body tumbles and oscillates.
Figure 7.6: Numerical integration for the three-link hydrodynamically decoupled model. Initial condition is $\eta_0 = [t \ 0 \ 0.01 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0]^T$. Integration is for time from 0 to 40, with parameter values $b = 0.2$ for all cases and $k_1 = k_2 = 0.4$ (top row), 0.8 (middle row) and 1.2 (bottom row). The left plots show trajectories of the center $C_2$ of $B_2$ in inertial frame in dashed line ($x$ scale is $[-6.5 \ 40.5]$), with snapshots of the fish at time $t = 0, 20$ and 40 overlaid. The right plots show orientation $\beta_2$ as functions of time. Only the $k_1 = k_2 = 0.8$ case (middle row) is stable.

### 7.2 2D stability analysis for the hydrodynamically coupled model

Now we turn to the hydrodynamically coupled case. The most important difference from the decoupled case is the calculation of the added mass: the decoupled case assumes a simplified (not accurate) form of the added mass, which yields analytical solution; the coupled case does not make any assumption, and numerically solve the potential. Besides this difference, the coupled case is very similar to the decoupled case, that is, the governing equation (7.1), relative equilibrium $\eta_e$, and the procedure of numerically evaluating Jacobian are all in the identical form of the decoupled case. Note that the numerical approach of linearization may be the only feasible method in linearizing the system since the added mass itself must be computed numerically.

One can compute the added mass for the $N$-link model in 2D in perfect fluid using a panel method, see e.g. [78] and [47]. The general idea of a penal method is
to approximate the body surface by a collection of “panels” (straight line segments in 2D or polygons in 3D), each with a singularity of source and/or vortex doublet of certain strength in order to satisfy the impermeability boundary condition on the surface. It can be implemented up to any desired degree of accuracy with the obvious expenses of computation time. In this work, since the fluid is irrotational, vortex singularity is not needed, only panels with source singularity will be used. The procedure is briefly described as follows: the surface of each ellipse is discretized into \( \mathcal{M} \) panels with the tips of ellipse having the most dense discretization, and the articulated body is discretized into \( N \mathcal{M} \) panels. Each panel is assumed to have a constant source distribution with a certain strength \( \sigma_i \). A matrix of “influence coefficients” \( \{a_{ij}\} \), i.e. the contribution of normal velocity component at the \( i^{th} \) panel from the \( j^{th} \) panel, is constructed. The impermeability boundary condition is implemented such that if the normal velocity at the \( i^{th} \) panel is denoted \( v_n^i \), the equation to determine the source strength of each panel is written as \( \{a_{ij}\}\{\sigma_i\} = \{v_n^i\} \). After solving for \( \sigma_i \), the potential \( \phi \) at the boundary of the body can be calculated, therefore the added mass \( M_F^F \) can be readily calculated.

**Two-link model** Since the procedure of calculating the eigenvalues associated with the Jacobian is essentially the same with the decoupled case (only the dependence of \( \theta_1 \) now also comes through the computation of added mass in \( M_{total} \)), we will directly show the stability result in the following paragraphs. As depicted in Figure 7.7, after a full parametric analysis, two regions in the \((b, k_1)\) parameter space can be readily identified for which all the eigenvalues have zero real part,
Figure 7.7: Stable regions of the two-link hydrodynamically coupled model in 2D in the $(b, k_1)$ parameter space. Two stable regions are found: one is the half plane $b \geq 1.68$, the other is the shark fin shaped area in $b < 1$. The dashed line region indicates the results from the hydrodynamically decoupled case.

which implies that the system is linearly stable for these regions of parameter: one is the half plane $b \geq 1.68$, the other is the shark fin shaped area in $b < 1$. Note the first region is different than the decoupled case, i.e. it is no longer simply $b \geq 1$. This should also be understood in analogy to the single ellipse. For a single ellipse, the coast motion is stable when $b \geq 1$, which means a single body is stable when it is in circular shape ($b = 1$), or when it is in oval shape ($b > 1$) and translating along its now semi-minor axis ($a$). For the two-link coupled model, when $b \approx 1.68$, the two ellipses combined with their added masses is similar to a circular shaped body, hence it is stable. When $b > 1.68$, the two-link fish model is similar to the oval shaped fish moving along its minor axis of symmetry. On the other hand, the second region is similar to the decoupled case, that is, by allowing an internal shape change via $\theta_1$, the two-link coupled model also shows that linear stability can be achieved for $b < 1$ given an appropriate elasticity. The dashed line indicates hydrodynamically decoupled shark fin region. One can see that although the decoupled
stable region does not completely coincide with the coupled region, it is a good approximation.

Figure 7.8: Schematics of the dominant stable mode of oscillation for the two-link hydrodynamically coupled model. Parameters are \( b = 0.1, k_1 = 0.35 \).

One can also obtain the modes of vibration in the shark fin shaped stable region. As mentioned before, four eigenvalues of \( A \), say, \( \lambda_i, i = 1, \ldots, 4 \) are zero. The other four eigenvalues, when in the stable region, are two pairs of pure imaginary complex conjugates, \( \lambda_{5,6} = \pm \mu_1 i \) and \( \lambda_{7,8} = \pm \mu_2 i \) (without loss of generality, assuming \( \mu_1 \leq \mu_2 \)), corresponding to two pairs of complex conjugate eigenvectors \( \xi_{1R} \pm i \xi_{1I} \) and \( \xi_{2R} \pm i \xi_{2I} \), respectively. Here \( \mu_1 \) and \( \mu_2 \) are the natural frequencies of oscillation of the modes, which are the real and imaginary parts of the eigenvectors. Since \( \mu_1 \) is smaller, the corresponding modes \( \xi_{1R} \) and \( \xi_{1I} \) dominate the oscillation. For example, when \( b = 0.1 \) and \( k_1 = 0.35 \), in which the equilibrium is linearly stable, the smaller natural frequency of oscillation is \( \mu_1 = 0.205 \), and the modes are

\[
\xi_{1R} = [0 -0.884 -0.102 0.257 0 0 -0.192 -0.214]^T, \quad \xi_{1I} = [0 0 0.218 -0.114 0 0 0 0]^T.
\]

A schematics of the dominant stable modes is depicted in Figure 7.8. One can see that the two ellipses oscillate symmetrically relative to the center. One can readily verify that the dominant modes in this stable region are always symmetric like in Figure 7.8.

The linear results can be verified by numerical integration of the nonlinear equations with initial condition slightly perturbed from \( \eta_e \). More specifically, we integrate (7.1) from \( t = 0 \) to 40 with initial condition \( \eta_0 = [t 0 0 0.01 1 0 0 0]^T \), i.e.
Figure 7.9: Numerical integration of nonlinear equations for the two-link hydrodynamically coupled model in 2D. Initial condition is \( \eta_0 = [t \ 0 \ 0 \ 0.01 \ 1 \ 0 \ 0 \ 0]^T \). Integration is for time from 0 to 40, with parameter values \( b = 0.1 \) for all cases and \( k_1 = 0.2 \) (top row), 0.35 (middle row) and 0.5 (bottom row). The left plots show trajectories of the hinge joint in inertial frame in dashed line (x scale is \([-3.5 \ 41.5]\)), with snapshots of the body motion at time \( t = 0, 23 \) and 40 overlaid. The right plots show orientation \( \beta_1 \) as functions of time. Only the \( k_1 = 0.35 \) case (middle row) is stable.

![Figure 7.9: Numerical integration of nonlinear equations for the two-link hydrodynamically coupled model in 2D.](image)

Figure 7.10: Energy of the two-link hydrodynamically coupled model in 2D. Kinetic energy of the bodies \( T_B = \sum_{i=1}^{2} T_{\text{B}_i} \), kinetic energy of the fluid \( T_F \), potential energy \( U \) and the total energy \( E \) are plotted as functions of time for three cases \( b = 0.1 \) and \( k_1 = 0.2 \) (unstable), 0.35 (stable), 0.5 (unstable).

![Figure 7.10: Energy of the two-link hydrodynamically coupled model in 2D.](image)

\( \theta_1 \) slightly perturbed from 0, for parameters \( b = 0.1 \) and \( k_1 = 0.2, 0.35 \) and 0.5. The three values of \( k_1 \) correspond to the systems being unstable \( (k_1 = 0.2) \), stable \( (k_1 = 0.35) \), and unstable \( (k_1 = 0.5) \), as predicted by Figure 7.7. The results are depicted in Figure 7.9, where trajectories of the hinge joint in inertial frame and orientation \( \beta_1 \) as functions of time are shown. Snapshots of the body motion in inertial frame are also overlaid on top of the trajectories at \( t = 0, 23 \) and 40 for
all three cases. Clearly, the equilibrium is stable only when \( k_1 = 0.35 \), while for \( k_1 = 0.5 \) the body tumbles and for \( k_1 = 0.2 \) the body tumbles and oscillates. The same conclusion holds for simulations with all kinds of perturbations. Figure 7.10 shows energies (kinetic energy of the bodies \( T_B = \sum_{i=1}^{2} T_{B_i} \), kinetic energy of the fluid \( T_F \), potential energy \( U \), and the total energy \( E = T_B + T_F + U \) as functions of time for the same initial condition as above for all three cases. Total energy \( E \) is always constant because the system is conservative. Figure 7.10(b) shows that the kinetic energy and potential energy oscillate because of the initial perturbation, but the system is still stable \( (k_1 = 0.35) \). Figure 7.10(c) shows the \( k_1 = 0.5 \) case, in which the whole body tumbles 180 degree as shown the in the bottom right plot of Figure 7.9. In the beginning, orientation of the fish is \( \beta_1 = 0 \), and \( T_B \) is much larger than \( T_F \); around \( t = 22 \), \( \beta_1 \approx \pi/2 \), and the majority of kinetic energy of the system is now in the fluid \( T_F \); finally, when \( \beta_1 \approx \pi \), the system is similar to the initial state except now the fish is traveling tail ahead. During this process, the potential energy peaks when \( \beta_1 \approx \pi/4 \) and \( \beta_1 \approx 3\pi/4 \), which are instants when \( T_B \) and \( T_F \) intersect, this means the shape change via \( \theta_1 \) is most significant at these instants. But \( U \) is still smaller compared to the kinetic energy since the spring is stiff, hence the shape change is relatively small. Figure 7.10(a) shows the unstable case when the spring is too weak \( k_1 = 0.2 \). One can see that, in general, kinetic energy exchange between \( T_B \) and \( T_F \) as the whole body tumbles 180 degree, just like in Figure 7.10(c). But the shape oscillation is much larger now, therefore \( U \) oscillates much stronger, and the the energy exchanges between \( T_B \), \( T_F \) and \( U \) through a complex process.
Three-link model For the three-link hydrodynamically coupled model in 2D, Figure 7.11(a) is a schematics of the stable region inside the \((0, 1) \times (0, 1) \times (0, 1)\) cube in parameter space \((b, k_1, k_2)\). Two stable regions are found: a large shark fin shaped region close to the corner and a small region in the middle. Indeed, the large region extends beyond \(k_1 > 1\) and \(k_2 > 1\), which is not shown in the plot. To compare this stability result with the two-link case, we plot the cross section of the

![Figure 7.11](image)

(a) stable region in \((b, k_1, k_2)\)  
(b) stable region on cross section \(\mathcal{I}\) \((k_1 = k_2)\)

Figure 7.11: (a) Stable regions of the three-link hydrodynamically coupled model in 2D for the \((b, k_1, k_2)\) parameter space. (b) Stable regions on cross section \(\mathcal{I}\) inside (a). The dashed line region indicates hydrodynamically decoupled case.

![Figure 7.12](image)

(a) dominant mode in small region  
(b) dominant mode in large region

Figure 7.12: Dominant modes of vibration in the stable region for the three-link hydrodynamically coupled model in 2D. (a) Dominant mode in the small region, the parameter is \(b = 0.1, k_1 = k_2 = 0.275\). (b) Dominant mode in the big region, the parameter is \(b = 0.2, k_1 = k_2 = 0.8\).
plane \( I \) in which \( k_1 = k_2 \) in Figure 7.11(b). One identifies three stable regions in total, the shark fin shaped region and the small region depicted in Figure 7.11(a), and an additional region \( b \geq 3.23 \) which is outside the cube in Figure 7.11(a). The region \( b \geq 3.23 \) is analogue to the \( b \geq 1.68 \) region in the two-link case. The shark fin shaped region is also analogue to the shark fin shaped region in the two-link model, but the area is larger than the latter. However, the small region does not have a counterpart in the two-link model. To better understand this region, we plot the dominant vibration modes in both the regions in \( b < 1 \). Figure 7.12(b) illustrates the dominant mode in the shark fin shaped region, the two side ellipses \( B_1 \) and \( B_3 \) are oscillating symmetrically relative to \( B_2 \). One can find similarity between Figure 7.12(b) and Figure 7.8, that is the dominant mode of the articulated body shows mirror-symmetry relative to its geometric center. Figure 7.12(a) shows the dominant mode in the small region, \( B_1 \) and \( B_3 \) are oscillating anti-symmetrically relative to \( B_2 \), and the mode of the body resembles a traveling wave. This mode can not exist in the two-link model, since a traveling wave mode can only exist for bodies consisting of more than one relative angle.

Just like the two-link case, the linear stability results are verified by direct numerical integration of the nonlinear equations with initial condition slightly perturbed from \( \eta_2 \). Here we focus on the small region. Integration is from \( t = 0 \) to 80 with initial condition \( \eta_0 = [t \ 0 \ 0 \ 0.001 - 0.001 \ 1 \ 0 \ 0 \ 0 \ 0]^T \) and parameters \( b = 0.1 \) and \( k_1 = k_2 = 0.15, 0.275 \) and 0.4. The three values of stiffness correspond to values for which the systems is unstable (0.15), stable (0.275), and unstable (0.4).
The results are depicted in Figure 7.13, where the trajectory of the center \( C_2 \) of \( B_2 \) and orientation \( \beta_2 \) as functions of time are shown. Snapshots of the body motion are also overlaid on top of the trajectories. Clearly, the equilibrium is stable only when \( k_1 = k_2 = 0.275 \). Figure 7.14 shows energies as functions of time for the

![Figure 7.13](image)

**Figure 7.13:** Numerical integration of nonlinear equations for the three-link hydrodynamically coupled model in 2D. Initial condition is \( \eta_0 = [t \ 0 \ 0 \ 0.001 \ -0.001 \ 1 \ 0 \ 0 \ 0 \ 0]^T \). Integration is for time from 0 to 80, with parameter values \( b = 0.1 \) for all cases and \( k_1 = k_2 = 0.15 \) (top row), 0.275 (middle row) and 0.4 (bottom row). The left plots show trajectory of the center of \( B_2 \) in inertial frame in dashed line (\( x \) scale is \([-5.5 \ 41.5]\) (top row), \([4.5 \ 51.5]\) (middle row), and \([14.5 \ 61.5]\) (bottom row)) with snapshots of the body motion at various instants overlaid (\( t = 0, 40 \) and 80 (top row), \( t = 10, 30 \) and 50 (middle row), and \( t = 20, 43 \) and 80 (bottom row)). The right plots show orientation \( \beta_2 \) as functions of time. Only the case \( k_1 = k_2 = 0.275 \) (middle row) is stable.

![Figure 7.14](image)

**Figure 7.14:** Energy of the three-link hydrodynamically coupled model in 2D. Kinetic energy of the bodies \( T_B = \sum_{i=1}^{3} T_{B_i} \), kinetic energy of the fluid \( T_F \), potential energy \( U \) and the total energy \( E \) are plotted as functions of time for three cases \( b = 0.1 \) and \( k_1 = k_2 = 0.1 \) (unstable), 0.275 (stable), 0.4 (unstable).
all cases, total energy $E$ is constant. Figure 7.14(b) shows that the kinetic energy
and potential energy oscillate because of the initial perturbation, but the system
is still stable ($k_1 = k_2 = 0.275$). Obviously, the oscillation is much smaller than
the two-link case since the initial perturbation is much smaller (0.001 vs. 0.02).
Figure 7.14(c) shows the $k_1 = k_2 = 0.4$ case, and it can be understood similarly to
the two-link case, that the articulated body tumbles almost 180 degree as shown
the bottom right plot in Figure 7.13. Though the energy exchange is more compli-
cated than the two-link case, because the springs are softer ($k_1 = k_2 = 0.275$ vs.
$k_1 = 0.7$). Figure 7.14(a) shows an even more complicated unstable case, since the
springs are much softer $k_1 = k_2 = 0.1$. But a tumbling of 180 degree can still be
identified, which agrees with the upper right plot in Figure 7.13.
Chapter 8

3D Stability Analysis

Figure 8.1: Three-link model of the fish in 3D is re-plotted. See caption in Figure 6.1 for detail.

We now discuss the stability of the coast motion of fish in 3D. We limit ourselves to only considering the hydrodynamically decoupled case in this section, though a study of the coupled case can be conducted similarly to the 2D model via a panel method in 3D, and the results are expected to be similar to the decoupled case as seen in the 2D study. As mentioned before, one of the potential complexities in 3D is to represent the orientation of bodies with respect to inertial frame. This is typically done using three Euler angles, and caution should be taken here in order not to encounter the singularities during this process. In this work, since the system
can be reduced into the Kirchhoff-like equations, and the reconstruction equations are treated completely as a “post-processing”, the difficulty from the orientation can be temporarily avoided by using only body frame velocities in the stability analysis. Hence, it is more convenient to follow the Newtonian approach in this chapter. We show the result for the three-link model as an example of the general \(N\)-link case.

Three-link hydrodynamically decoupled model in 3D  The governing equations for the three-link hydrodynamically decoupled model in 3D are given in (6.36). The relative equilibrium is given by \(W_e = [V_{2x_e} V_{2y_e} V_{2z_e} \Omega_{2x_e} \Omega_{2y_e} \Omega_{2z_e} \theta_{1e} \dot{\theta}_{1e} \theta_{2e} \dot{\theta}_{2e}]^T = [V_e 0 0 0 0 0 0 0 0 0]^T\). This is the coast motion of fish with coast speed \(V_e\).

Similar to the 2D case, we first nondimensionalize the equations using length scale \(a\), mass scale \(\rho \pi a^3\), and time scale \(a/V_e\). All terms can be readily nondimensionalized accordingly, for instance, one gets

\[
\begin{align*}
\tilde{a} &= 1, \quad \tilde{b} = \frac{b}{a}, \quad \tilde{c} = \frac{c}{a}, \quad \tilde{l} = \frac{l}{a}, \quad \tilde{V}_i = \frac{V_i}{V_e}, \quad \tilde{V}_e = 1, \\
\tilde{m}^{ib} &= \frac{m^B}{\rho \pi a^3}, \quad \tilde{j}^{ib} = \frac{J^B}{\rho \pi a^5}, \quad \tilde{k}_i = \frac{k_i}{\rho \pi a^3 V_e^2},
\end{align*}
\]

the entries in added mass can be nondimensionalized similarly as the actual mass and inertia, and the angles are already in dimensionless form. One is reminded that the units of body density (or equivalently, fluid density, since it is neutrally buoyant) are different in 2D and 3D, namely

\[
[ho]_{R^2} = \frac{[kg]}{[meter]^2}, \quad [ho]_{R^3} = \frac{[kg]}{[meter]^3}.
\]
For simplicity, we drop the $\hat{\cdot}$ notation with the understanding that all variables in the 3D case are dimensionless hereafter.

**In-plane and out-of-plane motions** We analyze the stability of relative equilibrium $\mathbf{W}_e$ using linearization method. Given the dimensionless coast speed is $V_e = 1$, $\mathbf{W}_e$ is now $\mathbf{W}_e = [1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]^T$. To linearize (6.36) around $\mathbf{W}_e$, we write $\mathbf{W} = \mathbf{W}_e + \delta \mathbf{W}$. Substitute $\mathbf{W}$ into (6.36) and eliminate higher order terms, the linearized equations of motion can be expressed as

$$
\bar{C} \delta \dot{\mathbf{W}} = \bar{D} \delta \mathbf{W}, \quad \Rightarrow \quad \delta \dot{\mathbf{W}} = \bar{A} \delta \mathbf{W}, \quad \bar{A} = \bar{C}^{-1} \bar{D}.
$$

(8.1)

The Jacobian $\bar{A}$ is a $10 \times 10$ matrix, and the eigenvalues associated with $\bar{A}$ are not in pairs as in the 2D case since the variables are no longer canonical. The 10 variables can be separated into two groups: *in-plane* variables and *out-of-plane* variables. In-plane variables are the ones that variations in these variables will result in a motion of the articulated body moving within the shape change defined plane $\Sigma$, and they are $V_{2x}$ (traditionally referred to as “surge”), $V_{2y}$ (“slip”), $\Omega_{2z}$ (“yaw”), and the relative angles $\theta_i$ and $\dot{\theta}_i$ (shape change). Out-of-plane variables, on the other hand, are the ones that variations in these variables will result in a motion of the articulated body moving out of the $\Sigma$ plane, and they are $V_{2z}$ (“heave”), $\Omega_{2x}$ (“roll”) and $\Omega_{2y}$ (“pitch”). Denote $\mathbf{W}_{\text{in}} = [V_{2x} \ V_{2y} \ \Omega_{2z} \ \theta_1 \ \dot{\theta}_1 \ \theta_2 \ \dot{\theta}_2]^T$ and
\( \mathbf{W}_{\text{out}} = [V_{2z}, \Omega_{2x}, \Omega_{2y}]^T \), one can rewrite the linear equation (8.1) in the following form

\[
\begin{pmatrix}
\delta \mathbf{W}_{\text{out}} \\
\delta \mathbf{W}_{\text{in}}
\end{pmatrix} = A
\begin{pmatrix}
\delta \mathbf{W}_{\text{out}} \\
\delta \mathbf{W}_{\text{in}}
\end{pmatrix}.
\]

And it turns out that the in-plane and out-of-plane groups are linearly decoupled from each other, namely

\[
A = \begin{bmatrix}
A_{\text{out}} & 0 \\
0 & A_{\text{in}}
\end{bmatrix}.
\]

(8.2)

**Linear stability analysis**  The Jacobian associated with out-of-plane variables \( A_{\text{out}} \) is a \( 3 \times 3 \) matrix, it has 3 eigenvalues \( \lambda_i^{\text{out}}, i = 1, 2, 3 \). One eigenvalue, say, \( \lambda_1^{\text{out}} \) is always 0, which reflects the fact that the system has another family of relative equilibria that is different from what we are interested in, namely all components are 0 except \( \Omega_{2x} \in \mathbb{R} \). This is a relative equilibrium of a constant roll with no translational velocity and all relative angles remaining 0. The other two eigenvalues are given by

\[
\lambda_{2,3}^{\text{out}} = \pm \sqrt{\frac{(M_z - M_x)M_x}{(J_y + 4lM_z)M_z}}.
\]

For \( \lambda_{2,3}^{\text{out}} \) to be pure imaginary or zero, one needs

\[
M_z \leq M_x \iff m^B + M_z^F \leq m^B + M_x^F \iff M_z^F \leq M_x^F \iff c \geq 1,
\]

note that \( b \) does not have any limitation regarding stability analysis from the out-of-plane variables. This can be understood in analogy to the 2D case as well as the
single ellipsoid. However, since the two groups of variables are linearly independent, stability result in one of the groups should not affect the result in the other group, at least linearly.

Figure 8.2: Schematics of the stable regions of the three-link hydrodynamically decoupled model in 3D associated with the “in-plane” motion in a \((0 1) \times (0 1) \times (0 1)\) cube in \((b, c, (k \equiv k_1 = k_2))\) parameter space.

For the in-plane motion, linear equation is given by

\[
C_{in} \delta \dot{W}_{in} = D_{in} \delta W_{in}, \quad \Rightarrow \quad \delta \dot{W}_{in} = A_{in} \delta W_{in}, \quad A_{in} = C_{in}^{-1} D_{in}.
\]
where

\[
C_{in} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 3M_y & 0 & 0 & lM_y & 0 & -lM_y \\
0 & lM_y & J_z + 2l^2M_y & 0 & J_z + l^2M_y & 0 & 0 \\
0 & -lM_y & J_z + 2l^2M_y & 0 & 0 & 0 & J_z + l^2M_y \\
0 & 0 & J_z + 4l^2M_y & 0 & l^2M_y & 0 & l^2M_y \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0
\end{bmatrix},
\]

\[
D_{in} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -3M_x & 0 & -\gamma & 0 & -\gamma \\
0 & \gamma & \gamma - lM_y & -k_1 - \gamma & 0 & 0 & 0 \\
0 & \gamma & -\gamma + lM_y & 0 & 0 & -k_2 - \gamma & 0 \\
0 & \gamma & 0 & k_1 & -l\gamma & k_2 & l\gamma \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix},
\]

in which \(\gamma = M_x - M_y\). The Jacobian associated with in-plane variables \(A_{in}\) is a 7×7 matrix, it has 7 eigenvalues \(\lambda_i^{in}, i = 1, \cdots, 7\). One eigenvalue, say, \(\lambda_1^{in}\) is always 0, reflecting the fact that any value of coast speed \(V_c\) is still a relative equilibrium. The other six eigenvalues depend on these parameters: aspect ratio \(b\) and \(c\), and spring stiffness \(k_1\) and \(k_2\). For simplicity, we assume \(k_1 = k_2 \equiv k\), i.e. we explore a cross section \(\mathcal{I}\) in the four dimensional parameter space \((b, c, k_1, k_2)\), and \(\mathcal{I}\) is a
three dimensional parameter space \((b, c, k)\). Figure 8.2 shows the stable regions in the \((0 1) \times (0 1) \times (0 1)\) cube in \(I\). One can also show the cross sections of the stable region by setting one of the parameter to be constant. The result is depicted in Figure 8.3 for several cross sections: (a) \(c = 0.5\), (b) \(c = 1\), (c) \(c = 10\), (d) \(b = 0.2\), (e) \(b = 0.5\), (f) \(b = 0.8\). The top row shows the cross sections for constant \(c\).

![Graph showing stable regions for different parameters](image)

Figure 8.3: Cross sections of stable regions of the three-link hydrodynamically decoupled model in 3D associated with the “in-plane” motion in the \((b, c, k)\) parameter space, (a) \(c = 0.5\), (b) \(c = 1\), (c) \(c = 10\), (d) \(b = 0.2\), (e) \(b = 0.5\), (f) \(b = 0.8\).

One can see that as \(c\) increases, the shape of stable regions approaches the 2D case depicted in Figure 7.5. Indeed, one can imagine when \(c \rightarrow \infty\), the shape of stable regions should be identical to the 2D case, except the scale of \(k\) also increases as \(c\) increases, which is due to the different nondimensionalizations between 2D and 3D.
One identifies two stable regions: the half space $b \geq 1$ and the shark fin cylinder. The bottom row of Figure 8.3 shows the cross sections for constant $b$. One can see that when $b$ increases beyond a certain value (which can be seen from the top row plots that the upper bound of $b$ for the shark fin is less than 0.8), no stable region exists in the parameter space. Obviously, as soon as $b$ reaches and increases beyond 1, the system is always linearly stable. Another view of the cross sections is depicted in Figure 8.4, where the cross sections are $c = b$, $c = 2b$ and $c = 3b$. Therefore, for the in-plane variables, a linearly stable relative equilibrium requires the parameters to be inside either the half space $b \geq 1$ or the shark fin region.

![Figure 8.4: Cross sections of stable region of the three-link hydrodynamically decoupled model in 3D associated with the “in-plane” motion. The cross sections $c = b$, $c = 2b$ and $c = 3b$ are plotted.](image)

For the full system of 10 variables, the stable region should be obtained by the combination of the results from in-plane and out-of-plane analysis. The result is two regions: one is the quarter space $(b \geq 1) \cap (c \geq 1)$, which can be easily understood in analogy to the single ellipsoid; the other is the $c \geq 1$ part of the shark fin. In general, for the relative equilibrium to be linearly stable, the fish needs to be “tall”
(c ≥ 1), not too “fat” (in the shark fin region which is usually b < 0.8), and with appropriate muscle elasticity (k in the shark fin region).

**Numerical verification of the linear stability result**  One can verify the linear stability results by numerical integration of the nonlinear equations (6.33) with initial condition slightly perturbed from the relative equilibrium $W_e$. Since the out-of-plane stability is similar to the single ellipsoid case, and it is easy to verify, we will only show the in-plane stability result. Specifically, we integrate the nonlinear equations from $t = 0$ to $t = 50$ with initial condition $W_0 = [1 0 0 0 0 0.05 0 -0.05 0]^T$, i.e. the relative angles $\theta_1$ and $\theta_2$ slightly perturbed from 0. The parameters are set to be $c = 1.1$ ($c > 1$, therefore stable for out-of-plane perturbations), $b = 0.2$, and three distinct values of spring constants $k_1 = k_2$ are tested: 0.4, 0.7 and 1. Indeed, after the numerical integration of nonlinear equations, one obtains velocities of $B_2$ expressed in its body-fixed frame and relative angles as functions of time. One then needs to solve the reconstruction equations similar to the two-link case (6.25) in order to obtain positions and orientations of all three bodies expressed in inertial frame. The results are depicted in Figure 8.5, where the left column shows sketches of snapshots of fish body motion at three instants $t = 0, 25$ and 50, the center column shows $y$ component of trajectories of three body centers $y_1, y_2$ and $y_3$ as functions of time, and the right column shows relative angles $\theta_1$ and $\theta_2$ as functions of time. The three rows in Figure 8.5 correspond to results of three different parameter sets $k_1 = k_2 = 0.4, 0.7$ and 1. Clearly, only the case $k_1 = k_2 = 0.7$ is stable as predicted by the linear stability analysis, for 0.4, the body
tumbles and oscillates, while for 1, the body tumbles. Note that the way the body goes unstable when springs are too soft or too stiff is exactly the same with the 2D case. The linear stability result can be readily verified by integrating nonlinear equations for initial conditions with all kinds of perturbations.

Figure 8.5: Numerical integration for the three-link hydrodynamically decoupled model in 3D. Initial condition is \( \mathbf{Y}_0 = [1 \, 0 \, 0 \, 0 \, 0 \, 0 \, 0 \, 0.05 \, 0 \, -0.05 \, 0]^T \). Integration is for time from 0 to 50, with parameter values \( b = 0.2, c = 1.1 \) for all cases and \( k_1 = k_2 = 0.4 \) (top row), 0.7 (middle row) and 1 (bottom row). The left column shows snapshot sketches of the body motion at \( t = 0, 25 \) and 50 for all three cases. The middle column shows \( y \) component of trajectories of three body centers \( y_1, y_2 \) and \( y_3 \) in inertial frame as functions of time, where the horizontal axes are \( t \) in a [0 50] interval, and vertical axes are \( y \) in a [−5 5] interval. And the right column shows relative angles \( \theta_1 \) and \( \theta_2 \) as functions of time, horizontal scales are time in a [0 50] interval, and vertical scales are [−1 1]. Only the \( k_1 = k_2 = 0.7 \) cases (middle row) is stable.
The coast motion of fish has been generally believed to be intrinsically unstable, and fish fins are believed to be playing the stabilizing role during the motion. While the fish fins are certainly important to locomotions and stability controls, the effects of

Figure 9.1: Summary of stability results of 2D models.
other factors in fish swimming also need to be recognized and analyzed. This work shows that, among others, body shape change and muscle elasticity is a reason for the stable coast motion of fish observed in nature. Specifically, fish is modeled as an articulated body made of \( N \) identical rigid ellipses (in 2D) or ellipsoids (in 3D) submerged in a perfect fluid. These rigid links are connected via \( N - 1 \) hinge joints, and torsional springs are located at the joints in order to mimic the muscle elasticity. The determinant parameters are found to be the dimensionless aspect ratio of the fish body and the stiffness of fish muscle modeled by the torsional springs. The main stability results of 2D model are summarized in Figure 9.1, where the stable regions of the hydrodynamically coupled model (added mass computed accurately) are plotted in shaded areas, and the decoupled model (added mass of each body assumed to be not effected by the other bodies) are plotted in dashed area. Only stable regions in the \([0 1] \times [0 1]\) window in the parameter space is plotted here. One can draw the following conclusions based on these stability results.

1. The coast motion of fish is found to be stable for certain regions in \( b < 1 \) (which is always unstable for a single ellipse), and the main stable regions are in shark fin shape. This also suggests that fish can not be too “fat” because the stable region has a maximum value of \( b \). For a given \( b \) in the stable region, spring stiffness needs to be within a certain interval. If the springs are too stiff, the fish will tumble just like an elongated single body; if the springs are too soft, the fish will oscillate and tumble.
2. Stability results of the hydrodynamically coupled and decoupled models are found to agree, but not exactly overlapping. In general, the decoupled model predicts a larger shark fin region than the coupled model, and the region obtained in the coupled model is mostly a subset of the decoupled region. The small region observed in three-link coupled model does not exist in the decoupled case.

3. In comparison to the two-link model, the shark fin region obtained from the three-link model is larger, for both the coupled and decoupled cases. The ranges of stable spring stiffness for a given aspect ratio in the two models are also different.

4. Based on the coupled model, a small stable region is found in the three-link model in addition to the shark fin region, and it does not have a counterpart in the two-link model. This is because the dominant modes of oscillation in the shark fin regions of both the two-link and three-link models are found to be symmetric about the vertical axis of symmetry of the body, but the dominant mode in the small region is an anti-symmetric, traveling wave mode. This mode can not occur in the two-link model since it requires the number of relative angles that are parametrizing the shape of body to be more than 1.

5. Though not shown in Figure 9.1, another stable region exists for the models in both coupled and decoupled cases. For the decoupled cases, that region is the half plane $b > 1, k \in \mathbb{R}$, which is analogous to the stable region of a single
ellipse case \((b > 1)\) because the added mass of each body is now precisely the same with a single ellipse. For the coupled cases, the region is a subset of \(b > 1\) because the added mass now respects the boundary of the whole body; the values of lower bound in \(b\) correspond to the situation that the total mass (actual mass plus added mass) of the body is close to a circular or oval shape whose minor axis of symmetry is the coast direction.

Several questions for the 2D motion remain to be answered. For example, one might ask how the stable region changes as the number of links \(N\) increases. Preliminary results show that for a four-link hydrodynamically coupled model, the shark fin region is larger than the three-link model, and the small region corresponding to a traveling wave mode exists, the half plane region also exists, though no additional region is found in comparison to the three-link case. Based on these results, one seems to be able to conclude that the area of shark fin region increases as the number of link increases, though a more rigorous proof needs to be conducted. Also, one can imagine that when \(N \to \infty\), the discretized body approaches a continuously shape changing sheet. It is important to point out that the stability of such body reminds of the flutter of a flag in a breeze. There is a range of critical flow velocities that excite a resonant bending instability causing the flag to flutter or flap stably, see [75]. Obviously the difference between the passive fish and the flag is that the fish itself is moving while the flag is fixed at a given point in a cross-flow. The stability analysis of the flag problem is concerned with the interaction of the shape deformations with the surrounding fluid without accounting for the translational
and rotational motions present in the fish problem. Another concern about the finite-link fish model is that fluid travels through the cavates between the links, which is obviously not true in nature. One way to eliminate such phenomenon is to wrap elastic membrane tightly around the links as a model of the fish skin. Such an attempt is taken in the work of Kajtar & Monaghan [42, 43], where a fish model with skin is shown to be able to actively swim more efficiently than a model without skin in viscous flow. Another way is to model the fish as a continuous shape changing elastic body, which is the model used in Kanso [44]. Though no research among these related models has focused on the stability of coast motion in perfect fluid without vorticity. The role of vorticity is completely omitted in this study, but it is expected to affect the stability of fish swimming. In a perfect fluid, usually vorticity is modeled as concentrated in point vortices in the flow field, which can be already present in the ambient flow (may be generated by other fish) or generated by the fish itself. Although the interaction between an elongated body and point vortices has been studied in the past [46, 44], none of them addressed explicitly the problem of passive stability of the body that have bending resistance from shape change. These questions are important to the understanding of fish swimming, and they will certainly be viable future directions for the follow-up researches to this study.

We have also shown the stability of the hydrodynamically decoupled model in 3D. The dynamics of fish can be separated into two groups of motions, i.e. in-plane and out-of-plane motions. In-plane motion corresponds to surge, slip, yaw and the
shape change via relative angles. Out-of-plane motion corresponds to heave, roll and pitch. These two motions are linearly independent from each other, and their stability results can be understood in analogy to their 2D counterparts. In fact, in the limit $c \to \infty$, the three-link 3D model becomes a three-link 2D model, and the result agrees with the 2D result; in the limit $b \to \infty$, the 3D model essentially simplifies to a 2D single ellipse model, and the result also agrees with the 2D case. Specifically, the out-of-plane stability requires dimensionless aspect ratio in the vertical direction to be $c > 1$, and the in-plane stable regions are: the half plane $b > 1$, and the shark fin shaped region when $b < 1$. In nature, the tropical fish are examples of the species of fish that meets such stability criterions (“tall” and “thin”). Due to similarity between the 2D and 3D cases, although the study is only conducted in the hydrodynamically decoupled 3D model, one should expect similar result in the coupled 3D model just like the 2D case. An important issue to point out is that the in-plane and out-of-plane motions are linearly independent, hence the perturbation in one group does not excite perturbations in the other group. For instance, when perturbed *only* in the in-plane variables, the dynamics obtained
from the *nonlinear* integration will remain in plane, regardless of the stability of in-plane or out-of-plane motions. This is not to claim that the two groups are nonlinearly independent, as when perturbed *simultaneously* in both groups, even in the parameter regions such that the body is linearly stable in both groups, the result of nonlinear integration may still be unstable. Figure 9.2 shows snapshots of the motion of a three-link hydrodynamically decoupled fish in 3D. The parameters are \( b = 0.7, c = 1.1 \) and \( k_1 = k_2 = 0.7 \), i.e. identical to the “stable” case shown before, but now the initial perturbation is not only in relative angles (in-plane) but also in the pitch motion (out-of-plane). Clearly, the nonlinear motion is unstable. Hence, even for a fish that has parameters within the linear stable regions predicted by this model, additional stabilization mechanism may still be required. This may be one of the reasons that fish use their fins all the time to stabilize almost all of their motions. Fish do not need to be always passively stable, because crucial aspects to their survival such as maneuverability are then compromised. One of the potential future directions is to understand the stability/maneuverability of fish with non-coincident centers of mass and buoyancy, and one can speculate that body elasticity, ambient vorticity as well as fin movement will all play important roles in the motion.

We derived equations in both Newtonian and Lagrangian approaches for the \( N \)-link model in 2D and 3D. In general, one can choose to use either approach when studying this problem and the results should be the same. We show the two approaches here because we think it is equally important to illustrate the two
procedures: physically it is more straightforward to understand the problem from the Newtonian point of view, yet mathematically it is easier to study the problem using equations derived from the Lagrangian approach. The difficulty in Newtonian approach is to cancel the intermediate constraint force terms and to reconstruct the inertial frame motion after solving for the velocities. Though it may prove to be more useful only dealing with the velocity equations when studying stability about a relative equilibrium. The difficulty in Lagrangian approach is to properly choose the variables in inertial frame (especially the variables that represent orientation). Yet the procedure in this approach is systematic, therefore easier to generalize into more complicated system. It is also the easier approach when dealing with a hydrodynamically coupled model. In hydrodynamically decoupled setups, the body-fluid system is a Hamiltonian system. The Hamiltonian structure of a single ellipsoid has been constructed, see for example, Marsden et al.[68] and Leonard [58]. In these works, the authors also studied the nonlinear Lyapunov stability of the body-fluid system using an energy-Casimir method to construct the Lyapunov function. For the multi-link fish model, Lagrangian approach has been the main focus of previous studies [45, 55], and a complete Hamiltonian structure is still absent. In fact, we have already found the Hamiltonian structure of a multi-link hydrodynamically decoupled 3D fish model, and consequently a way of analyzing the nonlinear stability of the system in the Lyapunov sense, but this part of work can not be in this thesis due to time constraint. This will be addressed in a forthcoming work in the near future [41].
Bibliography


