Ch. 3: Forward and Inverse Kinematics

Recap: The Denavit-Hartenberg (DH) Convention

- Representing each individual homogeneous transformation as the product of four basic transformations:

\[
A_i = \text{Rot}_{\alpha_i} \text{Trans}_{d_i} \text{Trans}_{a_i} \text{Rot}_{\theta_i}
\]

\[
= \begin{bmatrix}
    c_{\alpha_i} & -s_{\alpha_i} & 0 & 0 \\
    s_{\alpha_i} & c_{\alpha_i} & 0 & 0 \\
    0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
    1 & 0 & 0 & a_i \\
    0 & 1 & 0 & 0 \\
    0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
    1 & 0 & 0 & 0 \\
    0 & 1 & 0 & 0 \\
    0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
    c_{\theta_i} & -s_{\theta_i} & 0 & 0 \\
    s_{\theta_i} & c_{\theta_i} & 0 & 0 \\
    0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

\[
= \begin{bmatrix}
    c_{\alpha_i} & -s_{\alpha_i} & 0 & 0 \\
    s_{\alpha_i} & c_{\alpha_i} & 0 & 0 \\
    0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
    1 & 0 & 0 & a_i c_{\alpha_i} \\
    0 & 1 & 0 & a_i s_{\alpha_i} \\
    0 & 0 & 1 & d_i \\
    0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
    1 & 0 & 0 & 0 \\
    0 & 1 & 0 & 0 \\
    0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

\[
= \begin{bmatrix}
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    s_{\alpha_i} & c_{\alpha_i} & 0 & 0 \\
    0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
    1 & 0 & 0 & a_i c_{\alpha_i} \\
    0 & 1 & 0 & a_i s_{\alpha_i} \\
    0 & 0 & 1 & d_i \\
    0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
    1 & 0 & 0 & 0 \\
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\end{bmatrix}
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    s_{\alpha_i} & c_{\alpha_i} & 0 & 0 \\
    0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
    1 & 0 & 0 & a_i c_{\alpha_i} \\
    0 & 1 & 0 & a_i s_{\alpha_i} \\
    0 & 0 & 1 & d_i \\
    0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
    1 & 0 & 0 & 0 \\
    0 & 1 & 0 & 0 \\
    0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 1 \\
\end{bmatrix}
\]
Recap: the physical basis for DH parameters

- $a_i$: link length, distance between the $o_0$ and $o_i$ (projected along $x_i$)
- $\alpha_i$: link twist angle between $z_0$ and $z_i$ (measured around $x_i$)
- $d_i$: link offset, distance between $o_0$ and $o_i$ (projected along $z_0$)
- $\theta_i$: joint angle, angle between $x_0$ and $x_i$ (measured around $z_0$)

General procedure for determining forward kinematics

1. Label joint axes as $z_0$, ..., $z_{n-1}$ (axis $z_i$ is joint axis for joint $i+1$)
2. Choose base frame: set $o_0$ on $z_0$ and choose $x_0$ and $y_0$ using right-handed convention
3. For $i=1:n-1$,
   i. Place $o_i$ where the normal to $z_i$ and $z_{i-1}$ intersects $z_i$. If $z_i$ intersects $z_{i-1}$, put $o_i$ at intersection. If $z_i$ and $z_{i-1}$ are parallel, place $o_i$ along $z_i$ such that $d_i=0$
   ii. $x_i$ is the common normal through $o_i$, or normal to the plane formed by $z_{i-1}$ and $z_i$ if the two intersect
   iii. Determine $y_i$ using right-handed convention
4. Place the tool frame: set $z_n$ parallel to $z_{n-1}$
5. For $i=1:n$, fill in the table of DH parameters
6. Form homogeneous transformation matrices, $A_i$
7. Create $T_{e0}$ that gives the position and orientation of the end-effector in the inertial frame
Example 4: cylindrical robot with spherical wrist

- 6DOF: need to assign seven coordinate frames
  - But we already did this for the previous two examples, so we can fill in the table of DH parameters:

<table>
<thead>
<tr>
<th>link</th>
<th>a_i</th>
<th>α_i</th>
<th>d_i</th>
<th>θ_i</th>
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</table>

\[ \begin{bmatrix}
    r_{11} & r_{12} & r_{13} & d_1 \\
    r_{21} & r_{22} & r_{23} & d_2 \\
    r_{31} & r_{32} & r_{33} & d_3 \\
    0 & 0 & 0 & 1
\end{bmatrix} \]

\[ T_0^6 = T_3^2 T_2^1 = \]

Example 4: cylindrical robot with spherical wrist

- Note that \( z_3 \) (axis for joint 4) is collinear with \( z_2 \) (axis for joint 3), thus we can make the following combination:

\[ r_{31} = c_1 c_2 c_3 d_1 - c_1 s_2 s_3 + s_1 s_2 c_4 \]
\[ r_{32} = s_1 c_2 c_3 d_1 - s_1 s_2 s_3 + c_1 s_2 c_4 \]
\[ r_{33} = -s_1 c_2 s_3 - c_1 s_2 c_4 \]
\[ r_{12} = -c_3 c_4 d_1 - s_3 s_2 s_3 - s_3 s_2 c_4 \]
\[ r_{13} = -c_3 s_4 d_1 + s_3 s_2 c_4 \]
\[ r_{23} = -c_4 d_2 - s_3 s_2 c_4 \]
\[ r_{22} = -s_3 s_2 s_3 + c_3 s_2 c_4 \]
\[ r_{21} = s_3 c_2 s_3 - s_3 s_2 c_4 \]
\[ r_{11} = c_3 c_4 d_1 + s_3 s_2 c_4 \]
\[ d_1 = s_1 c_1 d_2 + c_1 d_1 \]
\[ d_2 = -s_1 s_1 d_1 + d_1 + d_2 \]
Example 5: the Stanford manipulator

- 6DOF: need to assign seven coordinate frames:
  1. Choose \( z_0 \) axis (axis of rotation for joint 1, base frame)
  2. Choose \( z_1-z_5 \) axes (axes of rotation/translation for joints 2-6)
  3. Choose \( x_i \) axes
  4. Choose tool frame
  5. Fill in table of DH parameters:

<table>
<thead>
<tr>
<th>link</th>
<th>( a_i )</th>
<th>( \alpha_i )</th>
<th>( d_i )</th>
<th>( \theta_i )</th>
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Example 5: the Stanford manipulator

- Now determine the individual homogeneous transformations:

\[
\begin{align*}
A_1 &= \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, &
A_2 &= \begin{bmatrix} c_2 & 0 & -s_2 & 0 \\ s_2 & 0 & c_2 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, &
A_3 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{align*}
\]
Example 5: the Stanford manipulator

- Finally, combine to give the complete description of the forward kinematics:

\[
T_x = A_1 \cdots A_n = \begin{bmatrix}
    r_{11} & r_{12} & r_{13} & d_1 \\
    r_{21} & r_{22} & r_{23} & d_2 \\
    r_{31} & r_{32} & r_{33} & d_3 \\
    0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
\begin{align*}
    r_1 &= c_1(c_\theta c_\phi - s_\phi s_\theta) - s_\theta s_\phi c_\theta + c_\phi s_\theta \\
    r_2 &= c_1(c_\theta c_\phi - s_\phi s_\theta) - s_\theta s_\phi c_\theta + c_\phi s_\theta \\
    r_3 &= -s_1(c_\theta c_\phi - s_\phi s_\theta) - c_\phi s_\theta \\
    r_4 &= c_1(-c_\phi s_\theta + c_\phi s_\theta) + c_\phi s_\theta \\
    r_5 &= c_1(c_\phi s_\theta + c_\phi s_\theta) - s_\phi s_\theta \\
    r_6 &= \cdots
\end{align*}
\]

Example 6: the SCARA manipulator

- 4DOF: need to assign five coordinate frames:
  1. Choose $z_0$ axis (axis of rotation for joint 1, base frame)
  2. Choose $z_2$ axis (axes of rotation/translation for joints 2-4)
  3. Choose $x_3$ axes
  4. Choose tool frame
  5. Fill in table of DH parameters:

<table>
<thead>
<tr>
<th>link</th>
<th>$a_i$</th>
<th>$\alpha_i$</th>
<th>$d_i$</th>
<th>$\theta_i$</th>
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\[\text{Diagram of SCARA manipulator}\]
Example 6: the SCARA manipulator

- Now determine the individual homogeneous transformations:

\[
\begin{align*}
A_1 &= \begin{bmatrix}
  c_1 & -s_1 & 0 & a_1 c_1 \\
  s_1 & c_1 & 0 & a_1 s_1 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 1 & 0
\end{bmatrix}, \quad \vdots \quad A_6 = \begin{bmatrix}
  c_4 & -s_4 & 0 & 0 \\
  s_4 & c_4 & 0 & 0 \\
  0 & 0 & 0 & 1 \\
  0 & 0 & 0 & 1
\end{bmatrix}
\end{align*}
\]

\[
T_6^0 = A_1 \cdots A_6 = \begin{bmatrix}
  c_1 c_4 + s_1 s_4 & -c_1 s_4 + s_1 c_4 & 0 & a_1 c_1 + a_2 c_2 \\
  s_2 c_4 - c_2 s_4 & -s_2 s_4 - c_2 c_4 & 0 & a_2 s_1 + a_3 s_2 \\
  0 & 0 & -1 & -d_3 - d_4 \\
  0 & 0 & 0 & 1
\end{bmatrix}
\]

Forward kinematics of parallel manipulators

- Parallel manipulator: two or more series chains connect the end-effector to the base (closed-chain)
- Gruebler’s formula (3D):

\[
\text{#DOF} = 6(n_L - n_J) + \sum_{i=1}^{n_J} f_i
\]

\[
\text{#DOF for joint } i
\]

*excluding ground
Forward kinematics of parallel manipulators

- Example (2D):

![Diagram of a 2D parallel manipulator with joint parameters L1, L2, and θ.]

Inverse Kinematics

- Find the values of joint parameters that will put the tool frame at a desired position and orientation (within the workspace)
  - Given $H$:
    $$ H = \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix} \in SE(3) $$
Example: the Stanford manipulator

\[ c_1 (e_x c_x + s_x s_y) s_z + s_x c_z = 0 \]
\[ s_1 (e_x c_x + s_x s_y) s_z - s_x c_z = 0 \]

- For a given \( H \):
  
  \[ H = \begin{bmatrix} 0 & 1 & 0 & -0.154 \\ 0 & 0 & 1 & 0.763 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]

- Find \( \theta_1, \theta_2, d_3, \theta_4, \theta_5, \theta_6 \):
  
  \[ s_x d_3 + c_x d_4 + d_5 (c_x s_x + c_y s_z) + c_x s_z = 0.763 \]
  \[ c_x d_3 + d_4 (c_x c_x - s_x s_z) = 0 \]

- One solution: \( \theta_1 = \pi/2, \theta_2 = \pi/2, d_3 = 0.5, \theta_4 = \pi/2, \theta_5 = 0, \theta_6 = \pi/2 \)

Inverse Kinematics

- For the forward kinematics there is always a unique solution
- The inverse kinematics may or may not have a solution
Overview: kinematic decoupling

- Appropriate for systems that have an arm a wrist

- Now, origin of tool frame, \( o_6 \), is a distance \( d_6 \) translated along \( z_6 \) (since \( z_5 \) and \( z_6 \) are collinear)
Inverse position

- Now that we have $[x_c \ y_c \ z_c]^T$ we need to find $q_1, q_2, q_3$

Background: two argument atan

- We use atan2(·) instead of atan(·) to account for the full range of angular solutions
  - Called ‘four-quadrant’ arctan

$$\text{atan2}(y, x) = \begin{cases} 
\text{atan}(-y/x) & y < 0 \\
\pi - \text{atan}(-y/x) & y \geq 0, x < 0 \\
\text{atan}(y/x) & y \geq 0, x \geq 0 \\
\pi/2 & y > 0, x = 0 \\
\text{undefined} & y = 0, x = 0 
\end{cases}$$
Example: RRR manipulator

1. To solve for $\theta_1$, project the arm onto the $x_0, y_0$ plane

   $$\theta_1 = \text{atan2}(x_c, y_c)$$

Caveats: singular configurations, offsets

- If $x_c = y_c = 0$, $\theta_1$ is undefined
- If there is an offset, then we will have two solutions for $\theta_1$: left arm and right arm
Left arm and right arm solutions

- Left arm:
- Right arm:

Therefore there are in general two solutions for $\theta_1$

s for $\theta_2$:
Left arm and right arm solutions

- The two solutions for $\theta_3$ correspond to the elbow-down and elbow-up positions respectively.

RRR: Four total solutions

- In general, there will be a maximum of four solutions to the inverse position kinematics of an elbow manipulator.
  - Ex: PUMA
Example: RRP manipulator

- Spherical configuration

Next class...

- Complete the discussion of inverse kinematics
  - Inverse orientation
  - Introduction to other methods
- Introduction to velocity kinematics and the Jacobian