Ch. 3: Forward and Inverse Kinematics

Recap: The Denavit-Hartenberg (DH) Convention

- Representing each individual homogeneous transformation as the product of four basic transformations:

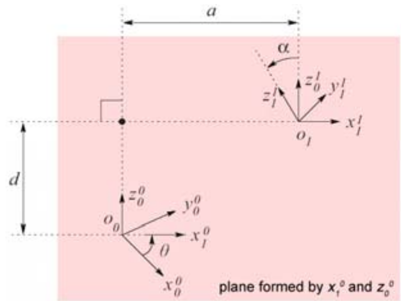
$$A_i = \mathbf{Rot}_{z,\theta_i}\mathbf{Trans}_{z,d_i}\mathbf{Trans}_{x,a_i}\mathbf{Rot}_{x,\alpha_i}$$

$$= \begin{bmatrix} c_{\theta_i} & -s_{\theta_i} & 0 & 0 \\ s_{\theta_i} & c_{\theta_i} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{\alpha_i} & -s_{\alpha_i} & 0 \\ 0 & s_{\alpha_i} & c_{\alpha_i} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c_{\theta_i} & -s_{\theta_i}c_{\alpha_i} & s_{\theta_i}s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i}c_{\alpha_i} & -c_{\theta_i}s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Recap: the physical basis for DH parameters

- a_i: link length, distance between the o_0 and o_1 (projected along x_1)
- α_i: link twist, angle between z_0 and z_1 (measured around x_1)
- d_i: link offset, distance between o_0 and o_1 (projected along z_0)
- θ_i: joint angle, angle between x_0 and x_1 (measured around z_0)

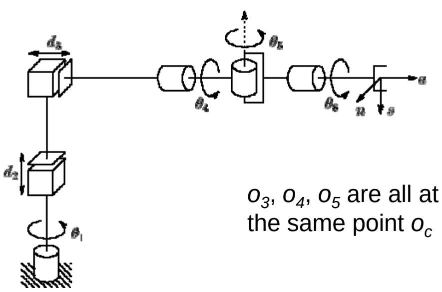


General procedure for determining forward kinematics

1. Label joint axes as z_0, …, z_{n-1} (axis z_i is joint axis for joint i+1)
2. Choose base frame: set o_0 on z_0 and choose x_0 and y_0 using right-handed convention
3. For i=1:n-1,
   i. Place o_i where the normal to z_i and z_{i-1} intersects z_i. If z_i intersects z_{i-1}, put o_i at intersection. If z_i and z_{i-1} are parallel, place o_i along z_i such that d_i=0
   ii. x_i is the common normal through o_i, or normal to the plane formed by z_{i-1} and z_i if the two intersect
   iii. Determine y_i using right-handed convention
4. Place the tool frame: set z_n parallel to z_{n-1}
5. For i=1:n, fill in the table of DH parameters
6. Form homogeneous transformation matrices, A_i
7. Create T_n^0 that gives the position and orientation of the end-effector in the inertial frame

Example 4: cylindrical robot with spherical wrist

- 6DOF: need to assign seven coordinate frames
  - But we already did this for the previous two examples, so we can fill in the table of DH parameters:



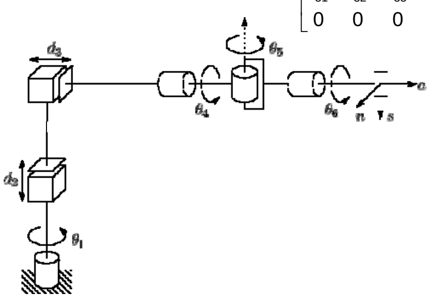
o_3, o_4, o_5 are all at the same point o_c

link	a_i	α_i	d_i	θ_i
1				
2				
3				
4				
5				
6				

Example 4: cylindrical robot with spherical wrist

- Note that z_3 (axis for joint 4) is collinear with z_2 (axis for joint 3), thus we can make the following combination:

$$T_6^0 = T_3^0 T_6^3 = \begin{bmatrix} r_{11} & r_{12} & r_{13} & d_x \\ r_{21} & r_{22} & r_{23} & d_y \\ r_{31} & r_{32} & r_{33} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

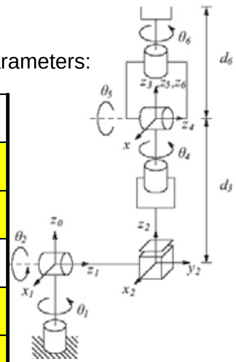


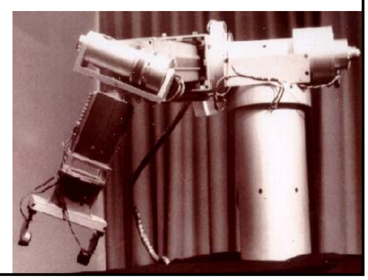
$$\begin{cases} r_{11} = c_1c_4c_5c_6 - c_1s_4s_6 + s_1s_5c_6 \\ r_{21} = s_1c_4c_5c_6 - s_1s_4s_6 - c_1s_5c_6 \\ r_{31} = -s_4c_5c_6 - c_4s_6 \\ r_{12} = -c_1c_4c_5s_6 - c_1s_4c_6 - s_1s_5c_6 \\ r_{22} = -s_1c_4c_5s_6 - s_1s_4s_6 + c_1s_5c_6 \\ r_{32} = s_4c_5c_6 - c_4c_6 \\ r_{13} = c_1c_4s_5 - s_1c_5 \\ r_{23} = \\ r_{33} = \\ d_x = \\ d_y = s_1c_4s_5d_6 + c_1c_5d_6 + c_1d_3 \\ d_z = -s_4s_5d_6 + d_1 + d_2 \end{cases}$$

Example 5: the Stanford manipulator

- 6DOF: need to assign seven coordinate frames:
  1. Choose z_0 axis (axis of rotation for joint 1, base frame)
  2. Choose z_1-z_5 axes (axes of rotation/translation for joints 2-6)
  3. Choose x_i axes
  4. Choose tool frame
  5. Fill in table of DH parameters:

link	a_i	α_i	d_i	θ_i
1				
2				
3				
4				
5				
6				





Example 5: the Stanford manipulator

- Now determine the individual homogeneous transformations:

$$A_1 = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} c_2 & 0 & s_2 & 0 \\ s_2 & 0 & -c_2 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

Example 5: the Stanford manipulator

- Finally, combine to give the complete description of the forward kinematics:

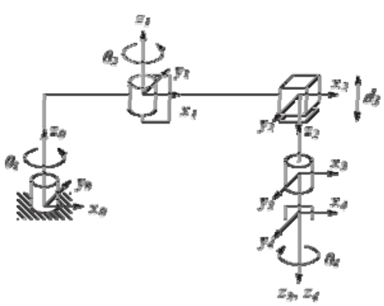
$$T_6^0 = A_1 \cdots A_6 = \begin{bmatrix} r_{11} & r_{12} & r_{13} & d_x \\ r_{21} & r_{22} & r_{23} & d_y \\ r_{31} & r_{32} & r_{33} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\left\{ \begin{aligned}
r_{11} &= c_1\left[c_2\left(c_4c_5c_6 - s_4s_6\right) - s_2s_5c_6\right] - d_2\left(s_4c_5c_6 + c_4s_6\right) \\
r_{21} &= s_1\left[c_2\left(c_4c_5c_6 - s_4s_6\right) - s_2s_5c_6\right] + c_1\left(s_4c_5c_6 + c_4s_6\right) \\
r_{31} &= -s_2\left(c_4c_5c_6 - s_4s_6\right) - c_2s_5c_6 \\
r_{12} &= c_1\left[-c_2\left(c_4c_5s_6 + s_4c_6\right) + s_2s_5s_6\right] - s_1\left(-s_4c_5s_6 + c_4c_6\right) \\
r_{22} &= -s_1\left[-c_2\left(c_4c_5s_6 - s_4c_6\right) - s_2s_5s_6\right] + c_1\left(-s_4c_5s_6 + c_4s_6\right) \\
r_{32} &= s_2\left(c_4c_5s_6 + s_4c_6\right) + c_2s_5s_6 \\
r_{13} &= c_1\left(c_2c_4s_5 + s_2c_5\right) - s_1s_4s_5 \\
r_{23} &= \\
r_{33} &= \\
d_x &= c_1s_2d_3 - s_1d_2 + d_6\left(c_1c_2c_4s_5 + c_1c_5s_2 - s_1s_4s_5\right) \\
d_y &= s_1s_2d_3 + c_1d_2 + d_6\left(c_1s_4s_5 + c_2c_4s_1s_5 + c_5s_1s_2\right) \\
d_z &=
\end{aligned} \right.$$

Example 6: the SCARA manipulator

- 4DOF: need to assign five coordinate frames:
  1. Choose z_0 axis (axis of rotation for joint 1, base frame)
  2. Choose z_1-z_3 axes (axes of rotation/translation for joints 2-4)
  3. Choose x_i axes
  4. Choose tool frame
  5. Fill in table of DH parameters:

link	a_i	α_i	d_i	θ_i
1				
2				
3				
4				



Example 6: the SCARA manipulator

- Now determine the individual homogeneous transformations:

$$A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & a_1c_1 \\ s_1 & c_1 & 0 & a_1s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} c_2 & s_2 & 0 & a_2c_2 \\ s_2 & -c_2 & 0 & a_2s_2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A_4 = \begin{bmatrix} c_4 & -s_4 & 0 & 0 \\ s_4 & c_4 & 0 & 0 \\ 0 & 0 & 1 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_4^0 = A_1 \cdots A_4 = \begin{bmatrix} c_{12}c_4 + s_{12}s_4 & -c_{12}s_4 + s_{12}c_4 & 0 & a_1c_1 + a_2c_{12} \\ s_{12}c_4 - c_{12}s_4 & -s_{12}s_4 - c_{12}c_4 & 0 & a_1s_1 + a_2s_{12} \\ 0 & 0 & -1 & -d_3 - d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Forward kinematics of parallel manipulators

- Parallel manipulator: two or more series chains connect the end-effector to the base (closed-chain)
- *Gruebler's formula* (3D):

$$\#\mathbf{DOF} = 6\left(n_L - n_j\right) + \sum_{i=1}^{n_j} f_i$$

n_L: number of links*

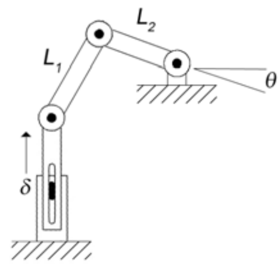
n_j: number of joints

f_i: #DOF for joint *i*

*excluding ground

Forward kinematics of parallel manipulators

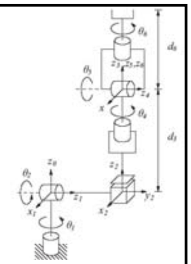
- Example (2D):



Inverse Kinematics

- Find the values of joint parameters that will put the tool frame at a desired position and orientation (within the workspace)
  - Given *H*:

$$H = \begin{bmatrix} R & o \\ 0 & 1 \end{bmatrix} \in SE(3)$$



Example: the Stanford manipulator

- For a given H:

$$H = \begin{bmatrix} 0 & 1 & 0 & -0.154 \\ 0 & 0 & 1 & 0.763 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned}
c_1[c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6] - d_2(s_4c_5c_6 + c_4s_6) &= 0 \\
s_1[c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6] + c_1(s_4c_5c_6 + c_4s_6) &= 0 \\
-s_2(c_4c_5c_6 - s_4s_6) - c_2s_5c_6 &= 1 \\
c_1[-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6] - s_1(-s_4c_5s_6 + c_4c_6) &= 1 \\
-s_1[-c_2(c_4c_5s_6 - s_4c_6) - s_2s_5s_6] + c_1(-s_4c_5s_6 + c_4c_6) &= 0 \\
s_2(c_4c_5s_6 + s_4c_6) + c_2s_5s_6 &= 0 \\
c_1(c_2c_4s_5 + s_2c_5) - s_1s_4s_5 &= 0
\end{aligned}$$

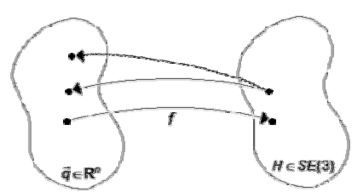
- Find θ_1, θ_2, d_3, θ_4, θ_5, θ_6:

$$\begin{aligned}
s_1s_2d_3 + c_1d_2 + d_6(c_1s_4s_5 + c_2c_4s_1s_5 + c_5s_1s_2) &= 0.763 \\
c_2d_3 + d_6(c_2c_5 - c_4s_2s_5) &= 0
\end{aligned}$$

- One solution: $\theta_1 = \pi/2$, $\theta_2 = \pi/2$, $d_3 = 0.5$, $\theta_4 = \pi/2$, $\theta_5 = 0$, $\theta_6 = \pi/2$

Inverse Kinematics

- For the forward kinematics there is always a unique solution
- The inverse kinematics may or may not have a solution

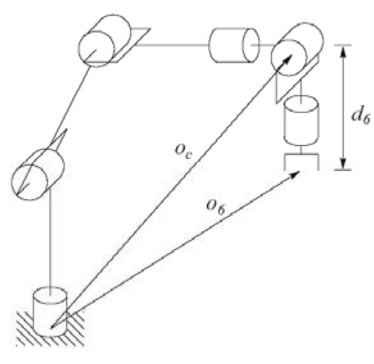


Overview: kinematic decoupling

- Appropriate for systems that have an arm a wrist

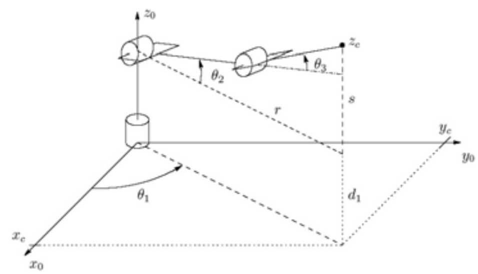
Overview: kinematic decoupling

- Now, origin of tool frame, o_6, is a distance d_6 translated along z_5 (since z_5 and z_6 are collinear)



Inverse position

- Now that we have $[x_c\ y_c\ z_c]^T$ we need to find q_1, q_2, q_3



Background: two argument atan

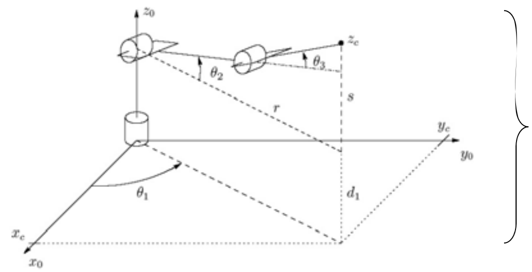
- We use atan2(·) instead of atan(·) to account for the full range of angular solutions
  - Called ‘four-quadrant’ arctan

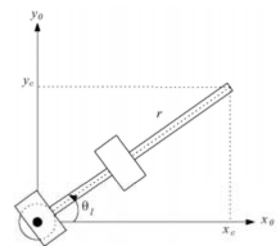
$$\mathbf{atan}2(y,x) = \begin{cases} -\mathbf{atan}2(-y,x) & y < 0 \\ \pi - \mathbf{atan}\left(-\frac{y}{x}\right) & y \geq 0, x < 0 \\ \mathbf{atan}\left(\frac{y}{x}\right) & y \geq 0, x \geq 0 \\ \frac{\pi}{2} & y > 0, x = 0 \\ \textbf{undefined} & y = 0, x = 0 \end{cases}$$

Example: RRR manipulator

1. To solve for θ_1, project the arm onto the x_0, y_0 plane

$$\theta_1 = \mathbf{atan}2(x_c, y_c)$$

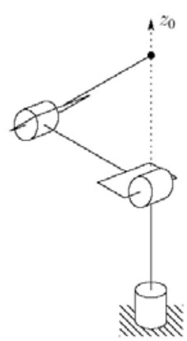


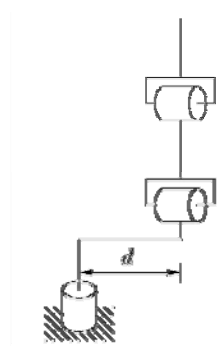


Caveats: singular configurations, offsets

- If x_c=y_c=0, θ_1 is undefined

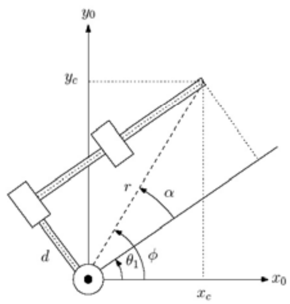
- If there is an offset, then we will have two solutions for θ_1: *left arm* and *right arm*

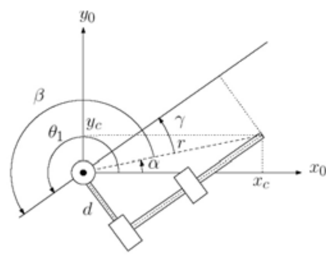




Left arm and right arm solutions

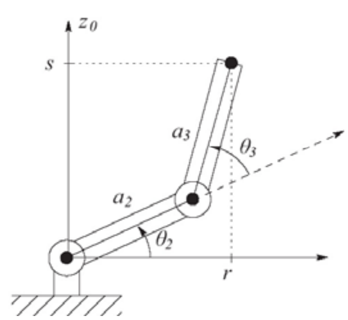
- Left arm:
- Right arm:





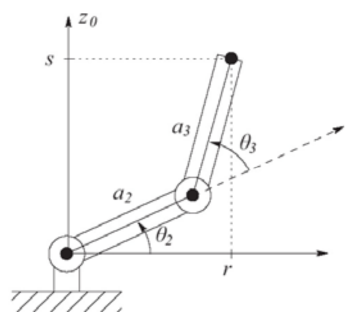
Left arm and right arm solutions

- Therefore there are in general two solutions for θ_1
- s for θ_3:



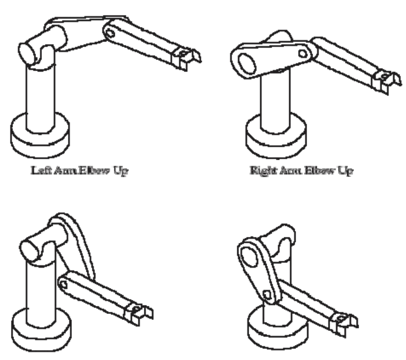
Left arm and right arm solutions

- The two solutions for θ_3 correspond to the elbow-down and elbow-up positions respectively



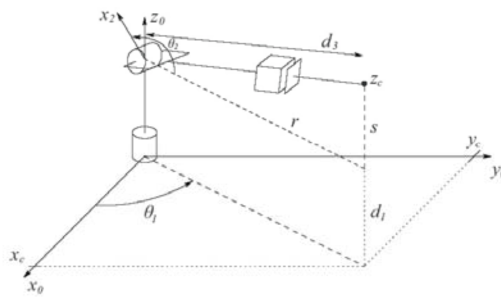
RRR: Four total solutions

- In general, there will be a maximum of four solutions to the inverse *position* kinematics of an elbow manipulator
  - Ex: PUMA



Example: RRP manipulator

- Spherical configuration



Next class…

- Complete the discussion of inverse kinematics
  - Inverse orientation
  - Introduction to other methods
- Introduction to velocity kinematics and the Jacobian