

The midterm may cover all of the material taught in class so far, including the current topic: NP-completeness. HW1-7.

Problem Even:

Input: n

Output: 1 if n is even and 0 otherwise.

The problem Even is polynomial-time reducible to 3-SAT. That means there exists an algorithm A such that A:

0 runs in polynomial time

1 on input n

- a) if n is even then outputs a 3-CNF formula Φ that is satisfiable.
- b) if n is odd then outputs a 3-CNF formula Φ that is not satisfiable.

Example A:

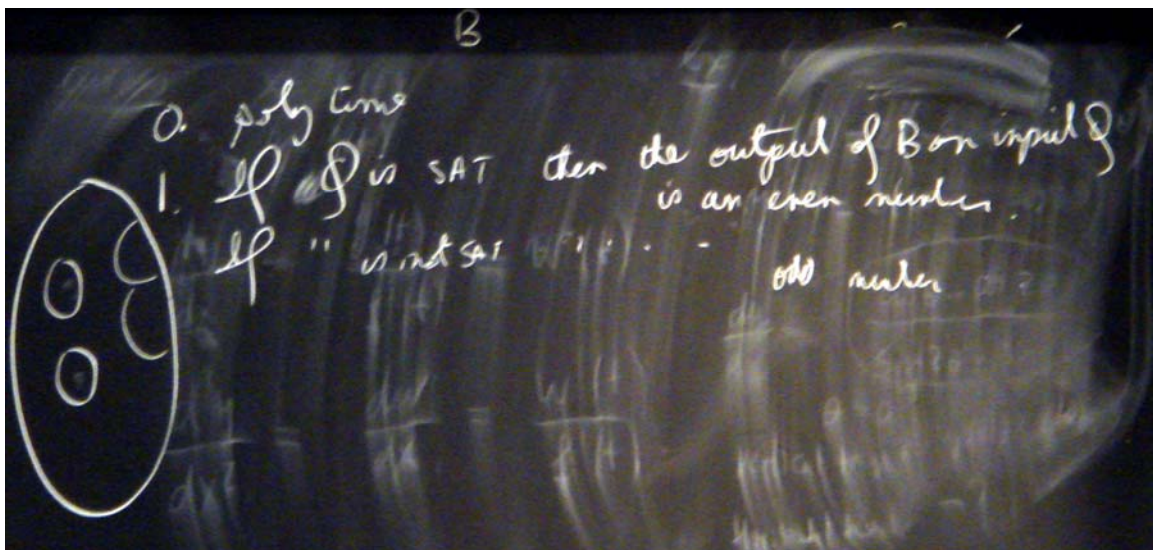
On input n : if n is even - output $(x_1 \vee x_1 \vee x_1)$; if n is odd - output $(x_1 \vee x_1 \vee x_1) \wedge (\neg x_1 \vee \neg x_1 \vee \neg x_1)$.

But 3-SAT is probably not polynomial-time reducible to Even. If it were, then 3-SAT would be polynomial-time solvable. That is, there probably isn't an algorithm B such that B:

0 runs in polynomial time

1 on input Φ

- a) if Φ is satisfiable outputs an even number.
- b) if Φ is not satisfiable outputs an odd number.



So, polynomial time reducibility is not a reflexive relation.

Theorem: The Clique problem is polynomial-time reducible to the Vertex Cover problem.

Example:



A complete graph is a graph with an edge between all possible pairs on nodes.



Observation: a graph on n nodes has a k -clique iff its complement has an $(n - k)$ -vertex cover.

Algorithm C:

Input G, k
Create the complement graph G'
 $k' = \#$ of nodes in $G - k$
Output G', k' .

C runs in polynomial time. We need to prove that if G has a k -clique then G' has a k' -vertex cover and if G does not have a k -clique then G' does not have a k' -vertex cover.



Algorithm D:

Input Φ

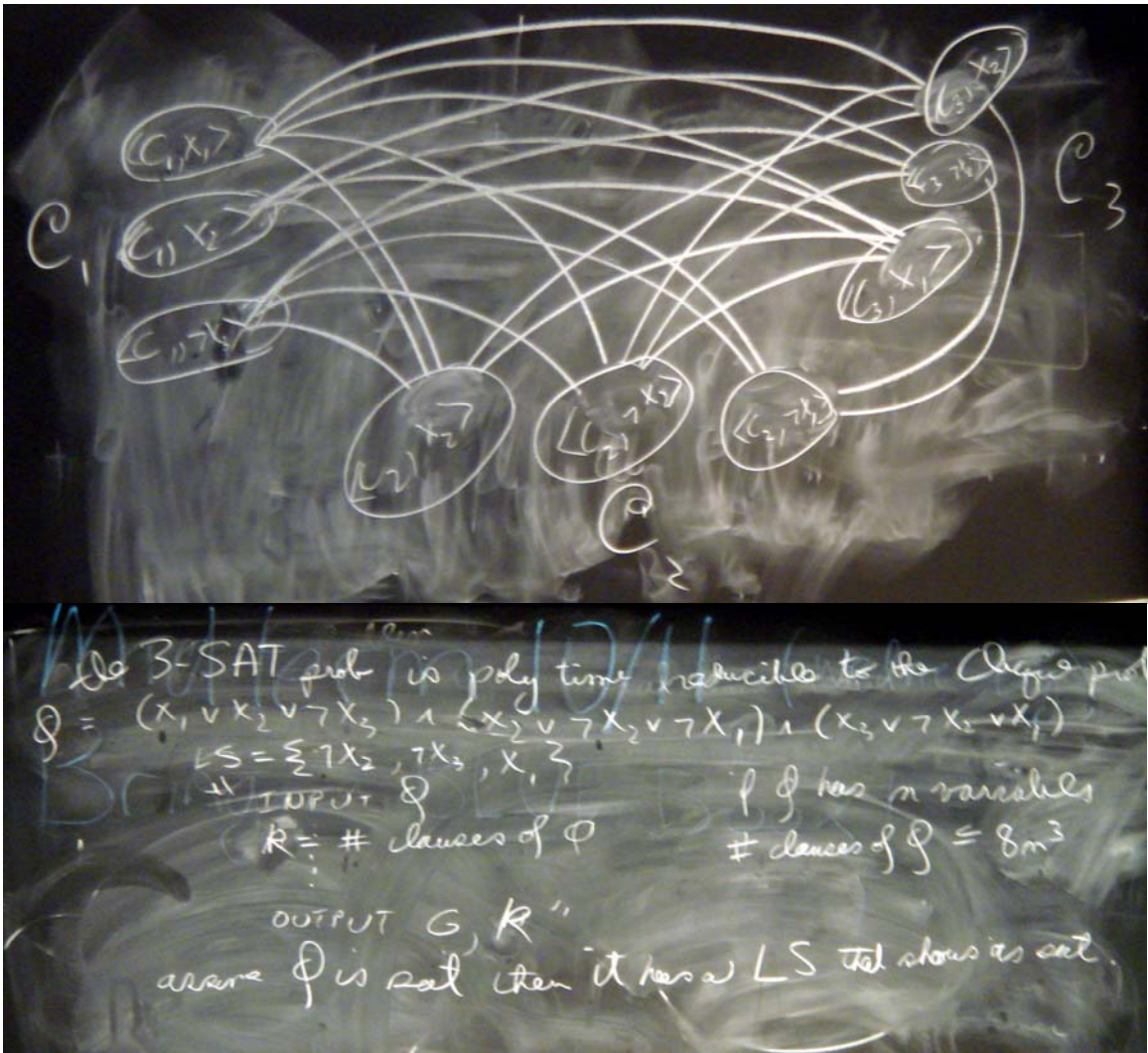
Create the graph G such that

for every clause C in Φ , there are 3 nodes in G , one for each literal in C

between every pair of nodes in **different** clauses and with non-conflicting literals (two literals are conflicting if one is the negation of the other; e.g. x_1 and $\neg x_1$) there is an edge.

$k = \#$ of clauses in Φ

Output G, k



D runs in polynomial time (there are no more than $8n^3$ clauses, so G has no more than $24n^3$ nodes and $(24n^3)^2 = 576n^6$ edges).

Assume Φ is satisfiable. Then it has a literal selection such that for every clause, at least one literal in that selection is in that clause. Take those k nodes. They form a k -clique because all non-conflicting literals have an edge between them.

Assume Φ is not satisfiable but G has a k -clique. Then that k -clique must have 1 node per clause (because the nodes within each clause have no edges between them) and no 2 will have a literal and its negation. Therefore you found a literal selection for Φ , so Φ must have been satisfiable. Contradiction #

Thus D is a polynomial-time reduction from 3-SAT to Clique.