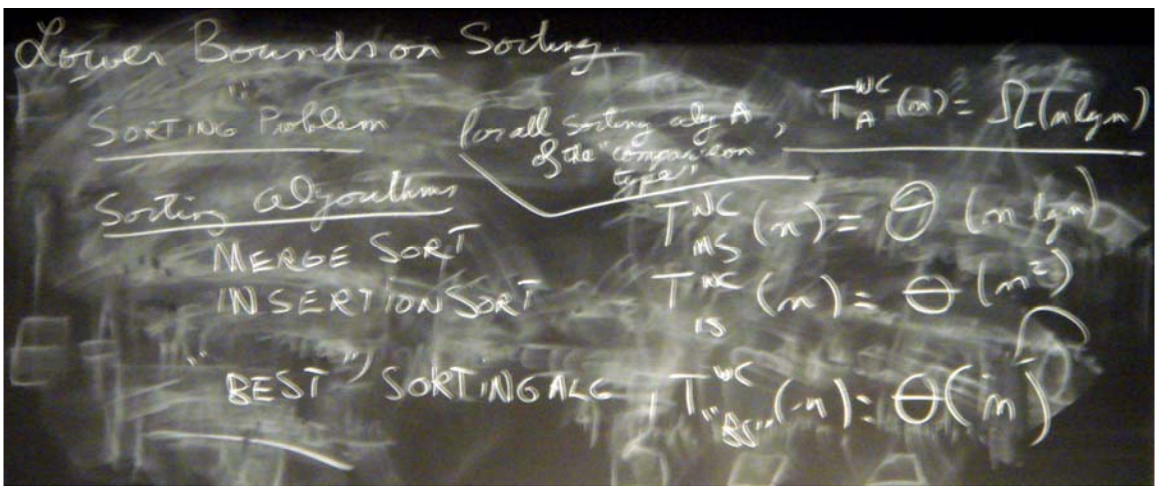
CSCI 303 Introduction to Algorithms
Spring 2007
February 5th, 2007 class notes

The first quiz will be on Wednesday, February 7th, 2007. BRING A BLUE BOOK!
The quiz will cover homeworks 1, 2 and 3.

Lower Bounds on Sorting:

Sorting problem:
Input: $a_1, \ldots, a_n$
Output: $a'_1, \ldots, a'_n$ such that these numbers are the input in sorted order

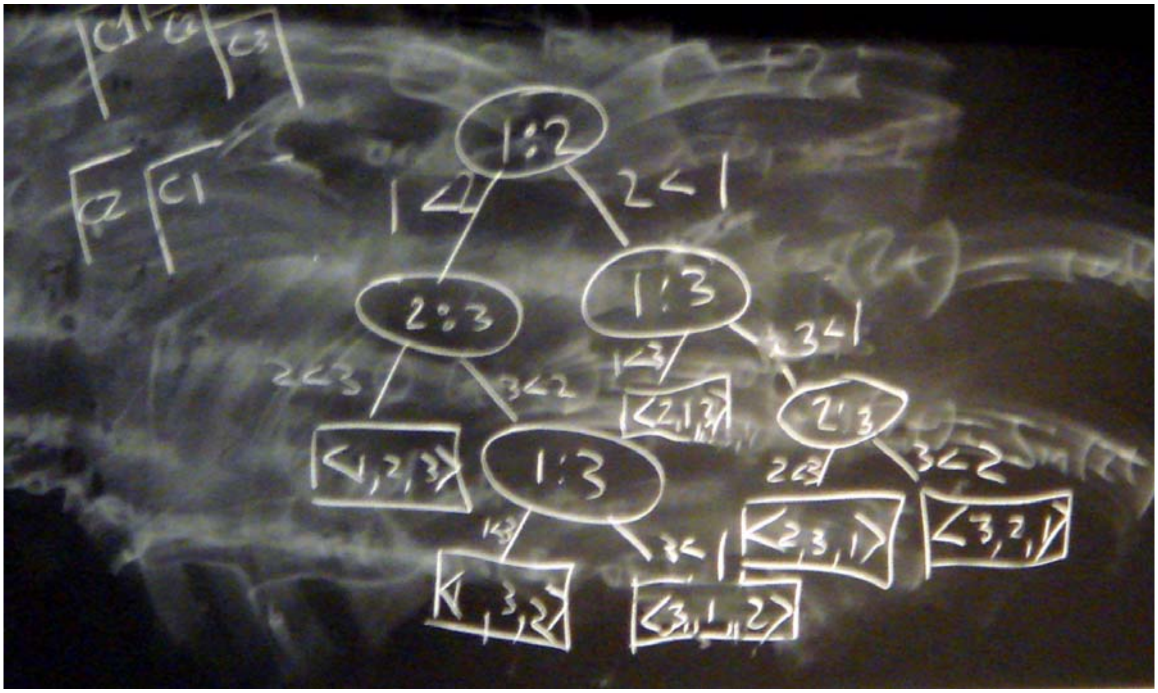
Sorting algorithms:
Insertion Sort: $T_{MS}^{WC}(n) = O(n^2)$
Merge Sort: $T_{IS}^{WC}(n) = \Theta(n \lg n)$



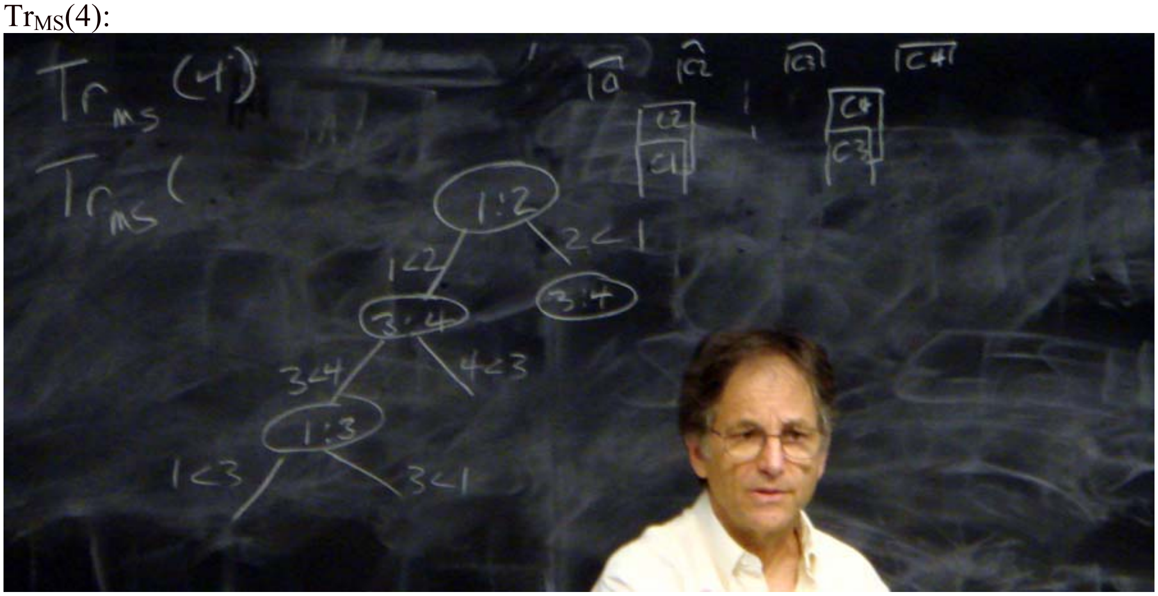
Theorem: For all "comparison" sorting algorithms A, $T_A^{WC}(n) = \Omega(n \lg n)$.

"Comparison" sorting algorithms are ones that can generate decision trees of the following types:

$Tr_{IS}(3)$:



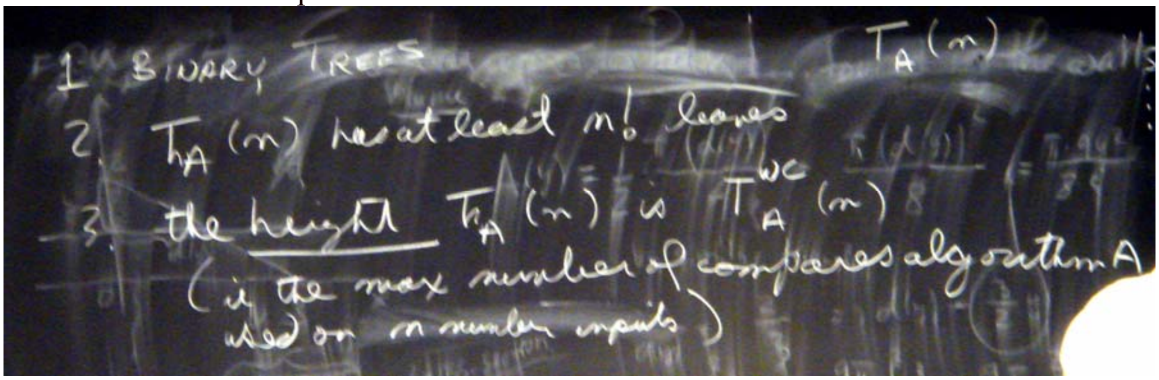
$Tr_{MS}(4)$:



1. All such trees are binary.

2. $Tr_A(n)$ has at least n! leaves

3. The height of $Tr_A(n)$ is $T_A^{WC}(n)$. I.e. the maximum number of comparisons A uses on n-number inputs.

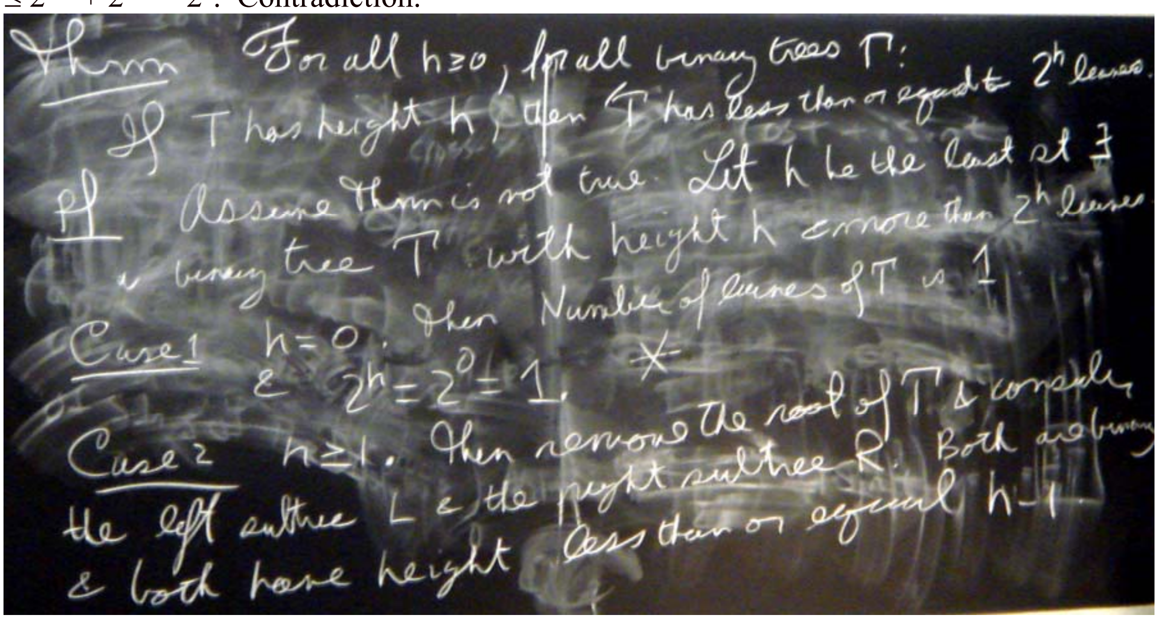


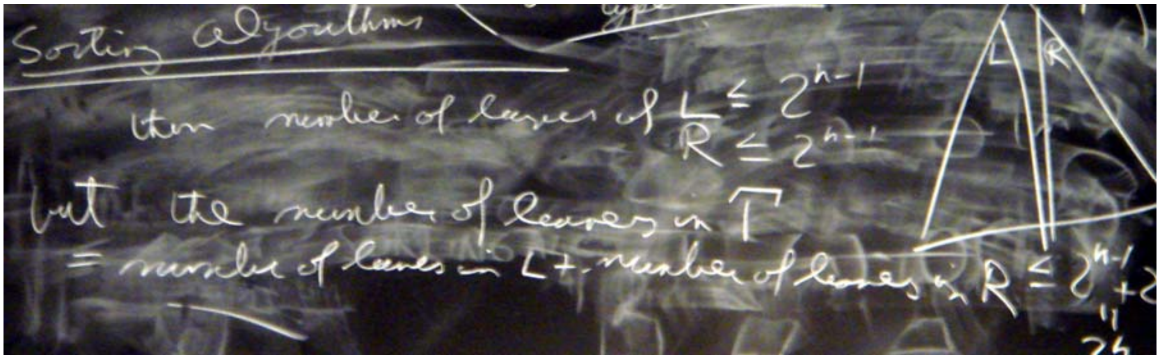
Theorem: For all h ≥ 0, for all binary trees T: if T has height h, then T has fewer than or equal to 2^h leaves.

Proof: Assume the theorem is not true. Let h be minimal such that there exists a tree T with height h and more than 2^h leaves.

Case 1: $h = 0$ Then the number of leaves is $1 = 2^0$. Contradiction.

Case 2: $h \geq 1$ Then remove the root of T and consider the left subtree L and right subtree R. Both are binary trees and have height less than or equal to $h - 1$. Then the number of leaves of L is $\leq 2^{h-1}$. Same for R. But T has only the leaves in L and R, which $\leq 2^{h-1} + 2^{h-1} = 2^h$. Contradiction.





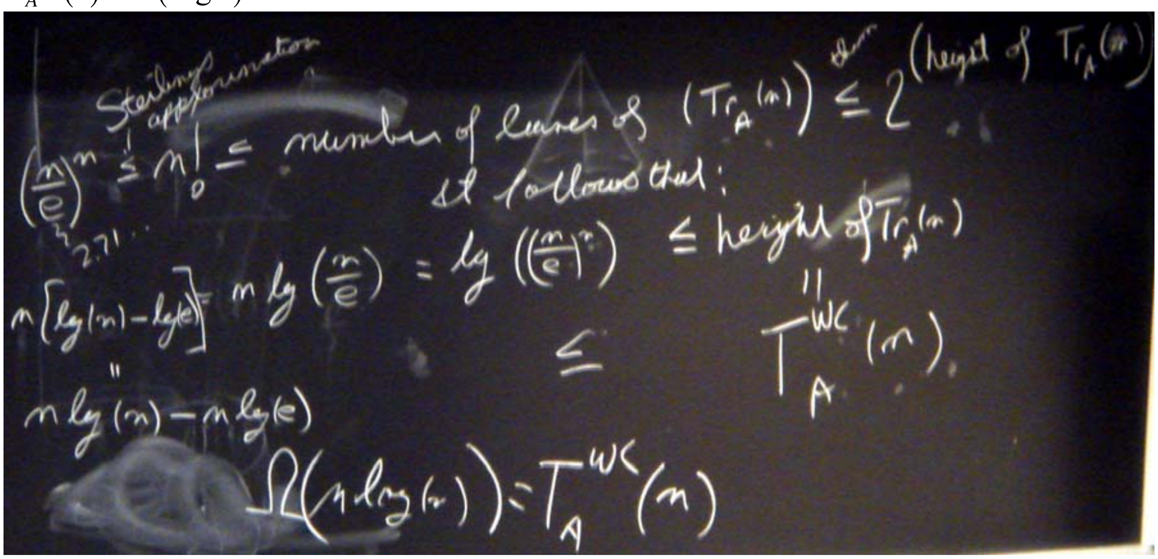
n! ≤ number of leaves of $Tr_A(n) \le 2^{height(Tr_A(n))}$

Sterling's approximation: $\left(\frac{n}{e}\right)^n \le n!$

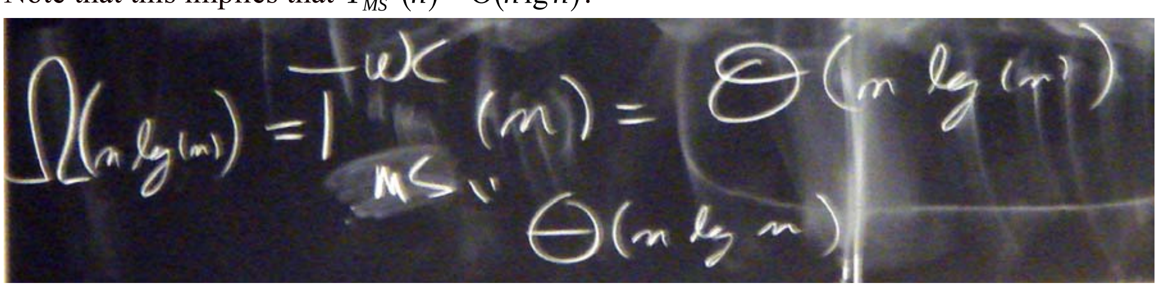
$height(Tr_A(n)) \ge \lg\left(\left(\frac{n}{e}\right)^n\right) = n\lg\left(\frac{n}{e}\right) = n\lg n - n\lg e$

So, $T_A^{WC}(n) \ge n\lg n - n\lg e$

$T_A^{WC}(n) = \Omega(n\lg n)$



Note that this implies that $T_{MS}^{WC}(n) = \Theta(n\lg n)$.



Thus we have shown the theorem: For all "comparison" sorting algorithms A, $T_A^{WC}(n) = \Omega(n\lg n)$.