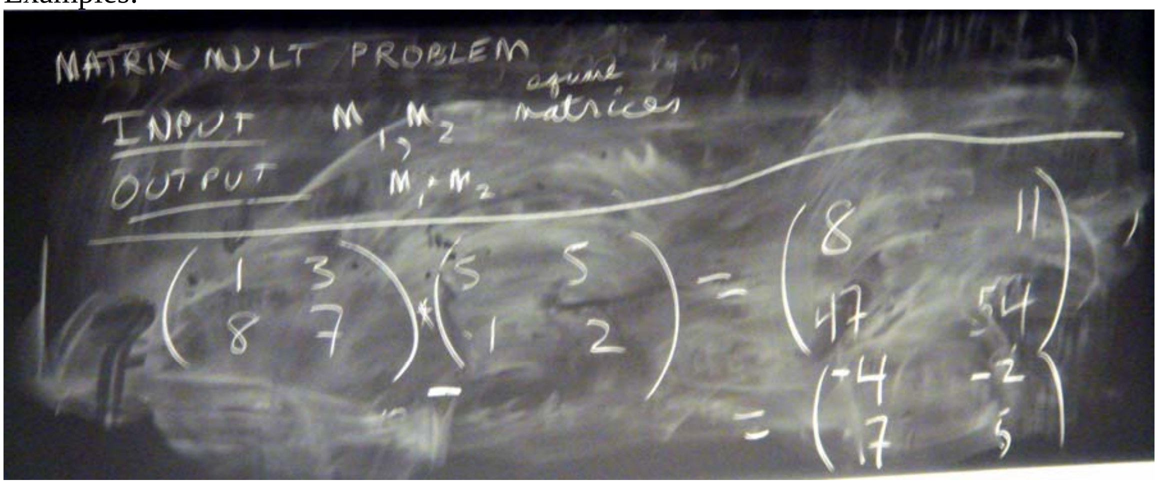
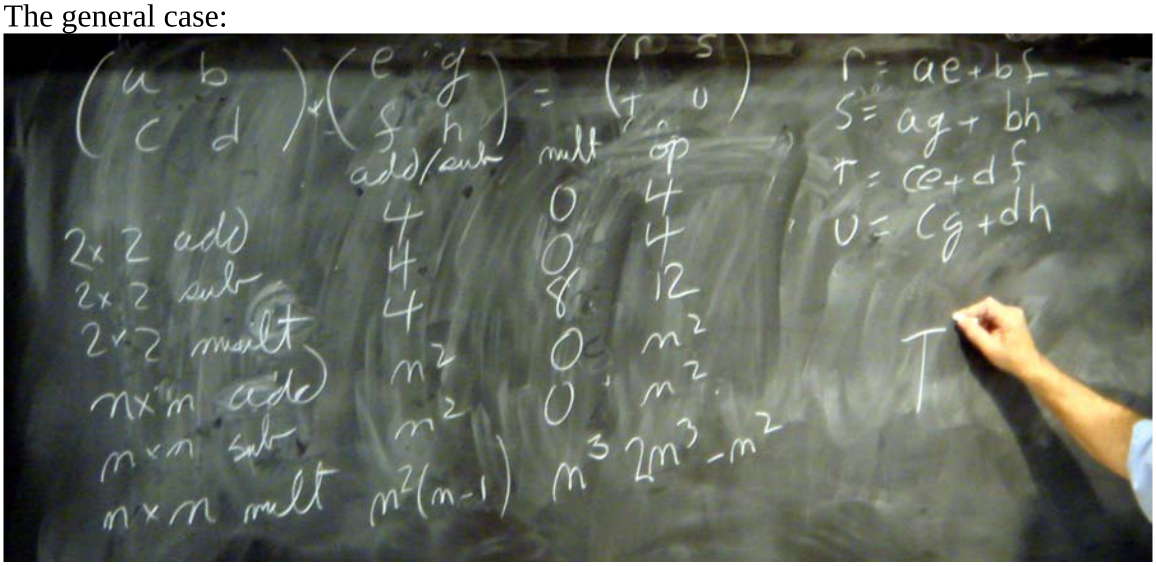
CSCI 303 Introduction to Algorithms
Spring 2007
January 24th, 2007 class notes

Matrix Multiplication problem:
Input: M_1, M_2 two n by n square matrices
Output: $M = M_1 M_2$

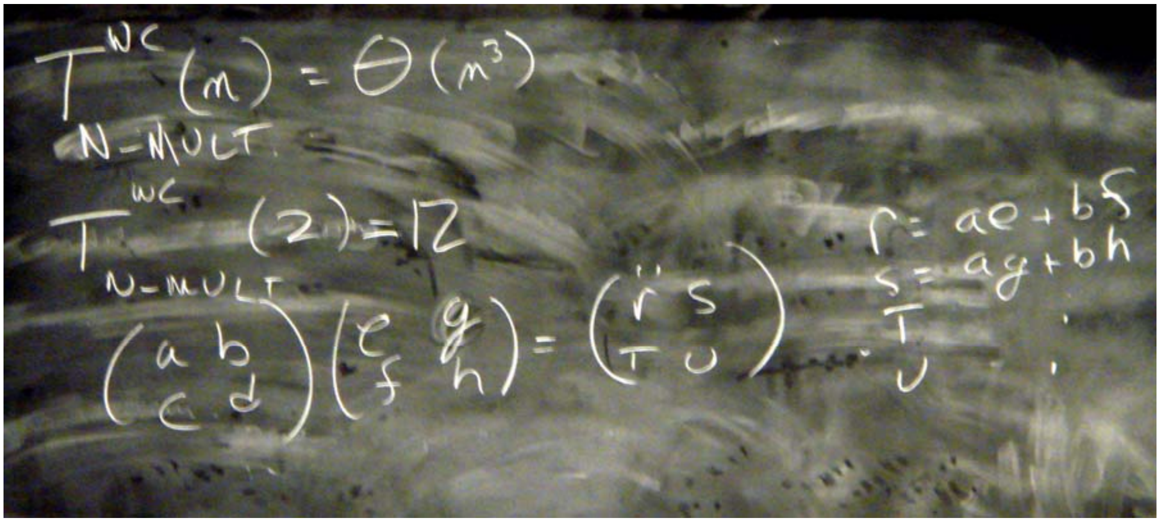
Examples:



The general case:



Thus, $T_{naiveMM}^{WC}(n) = \Theta(n^3)$.



Naïve matrix multiplication algorithm:

$T^{wc}_{N-Mult}(n) = \Theta(n^3)$

$T^{wc}_{N-Mult}(2) = 12$

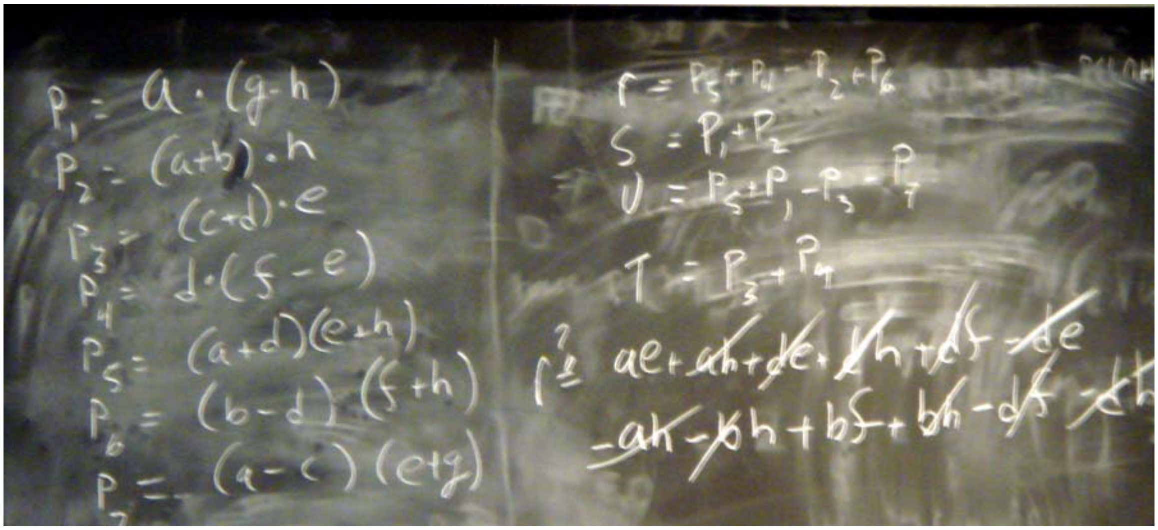
Strassen discovered that in multiplying 2 by 2 matrices

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

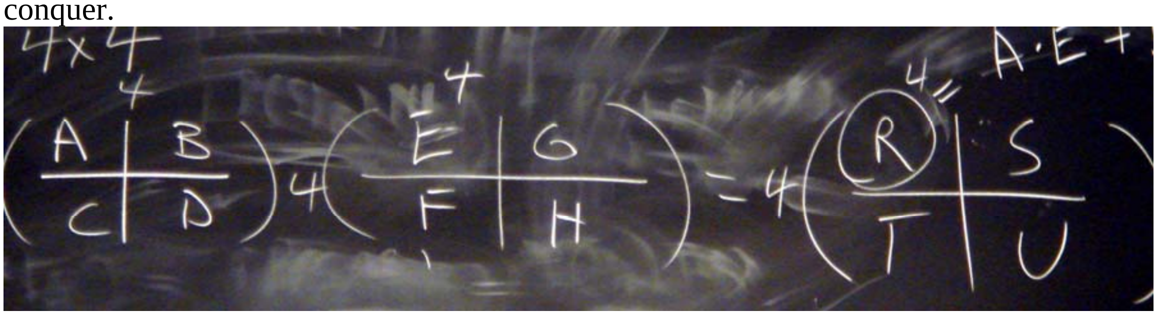
One can compute the entries by computing

$$P_1 = (A_{12} + A_{22})(B_{21} + B_{12})$$
$$P_2 = (A_{21} + A_{22})B_{11}$$
$$P_3 = A_{11}(B_{12} - B_{22})$$
$$P_4 = A_{22}(B_{21} - B_{11})$$
$$P_5 = (A_{11} - A_{12})B_{22}$$
$$P_6 = (A_{21} - A_{22})(B_{11} + B_{12})$$
$$P_7 = (A_{12} - A_{22})(B_{21} + B_{22})$$

$$C_{11} = P_1 + P_4 - P_5 + P_7$$
$$C_{12} = P_3 + P_5$$
$$C_{21} = P_2 + P_5$$
$$C_{22} = P_1 + P_3 - P_2 + P_6$$

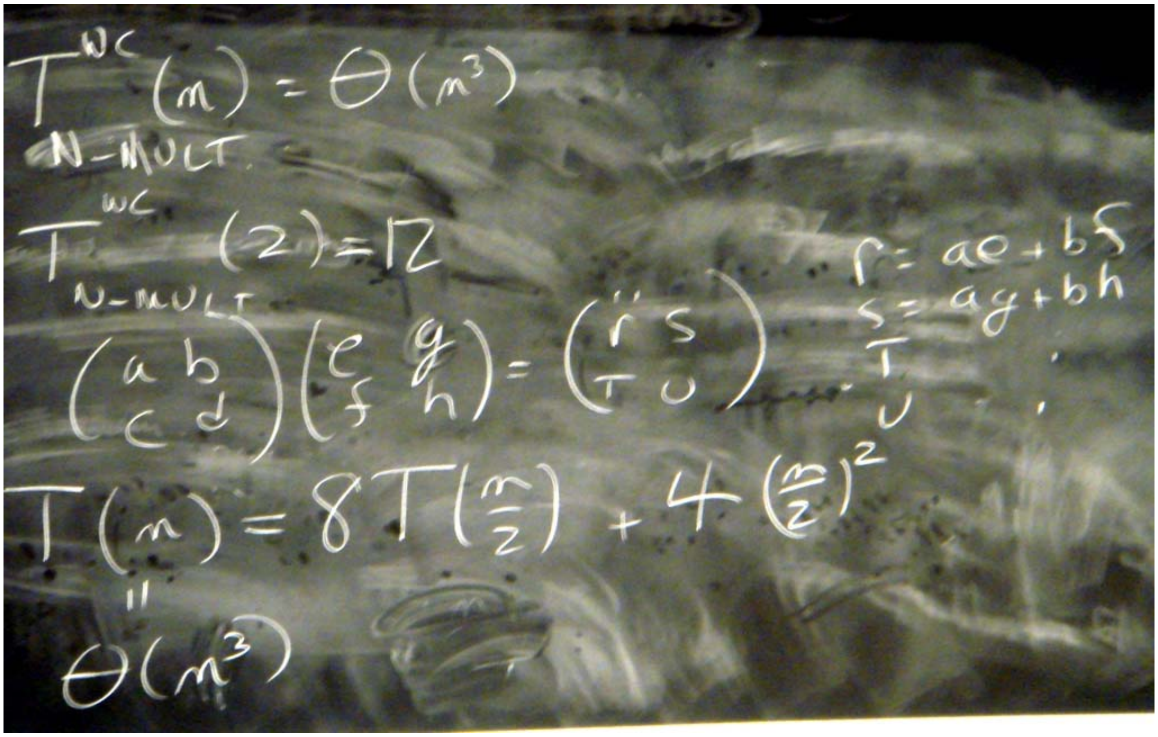


Thus Strassen's algorithm requires 18 additions / subtractions and 7 multiplications for a total of 25 operations. It can also, recursively, multiply larger matrices using divide and conquer.

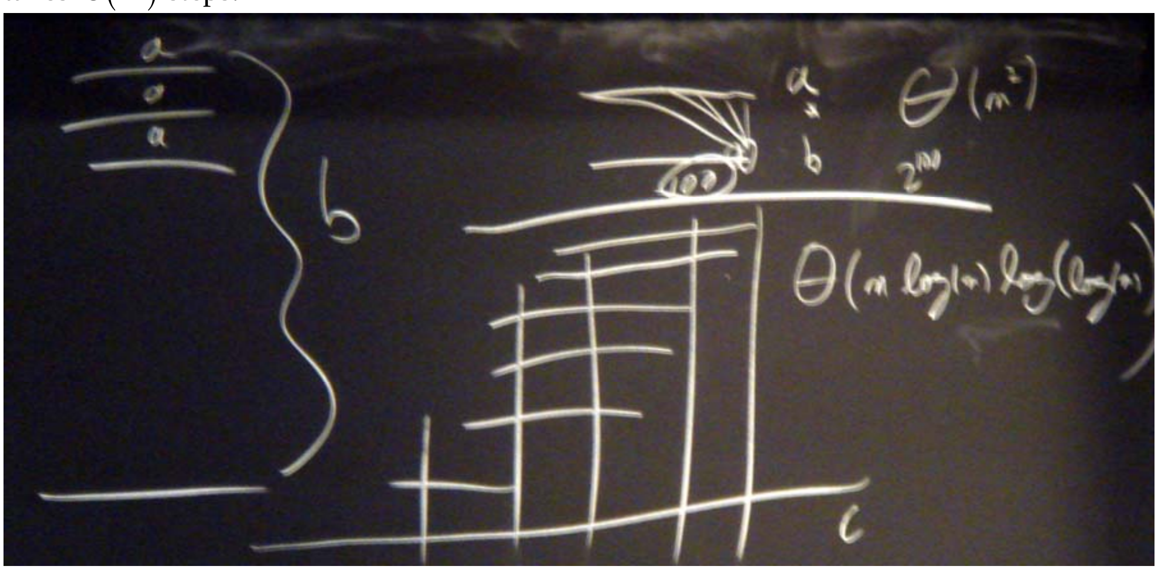


$$T_{N-Mult}^{wc}(n) = 8T_{N-Mult}^{wc}\left(\frac{n}{2}\right) + 4\left(\frac{n}{2}\right)^2 = \Theta\left(n^3\right)$$

$$T_{S-Mult}^{wc}(n) \leq 7T_{N-Mult}^{wc}\left(\frac{n}{2}\right) + 18\left(\frac{n}{2}\right)^2 = O\left(n^{\lg 7}\right) \approx O\left(n^{2.81}\right)$$



Strassen also can multiply n-bit numbers using $\Theta(n \lg n \lg \lg n)$ steps (the naïve approach takes $\Theta(n^2)$ steps.



<u>Order Statistics:</u>
Given a list of number $a_1, \ldots a_n$.
min: 1^{st} order statistic
2^{nd} smallest: 2^{nd} order statistic
i^{th} smallest: i^{th} order statistic
lower median: $\left\lfloor \frac{n}{2} \right\rfloor$ order statistic
upper median: $\left\lceil \frac{n}{2} \right\rceil$ order statistic

Minimum problem:
Input: $a_1, \ldots a_n$
Output: 1^{st} order statistic

There are similar problems for other order statistics.
$T_{N-MIN}^{wc}(n) = \Theta(n)$
$T_{N-MAX}^{wc}(n) = \Theta(n)$
$T_{N-2nd}^{wc}(n) = \Theta(n)$
$T_{N-43rd}^{wc}(n) = \Theta(n)$

Median problem:
Input: $a_1, \ldots a_n$
Output: lower median

$T_{Naive-Med}^{wc}(n) = \Theta(n^2)$
$T_{SORT-Med}^{wc}(n) = \Theta(n \lg n)$