HW 2 Solutions: sections 1.3-1.6

1 Section 1.3

Problem 8 Translate these statements into English, where $R(x)$ is "x is a rabbit" and $H(x)$ is "x hops" and the universe of discourse consists of all animals.

(a) $\forall x(R(x) \rightarrow H(x))$

(b) $\forall x(R(x) \wedge H(x))$

(c) $\exists x(R(x) \rightarrow H(x))$

(d) $\exists x(R(x) \wedge H(x))$

Solution Note that part(b) and part(c) are not the sorts of things one would normally say.

(a) If an animal is a rabbit, then that animal hops. (Alternatively, every rabbit hops.)

(b) Every animal is a rabbit and hops. (But this is a silly statement, don't you think?)

(c) There exists such an animal that if it is a rabbit then it hops. (Note that this is trivially true, satisfied, for example by lions, so it is not the sort of thing that one would say.)

(d) There exists one animal that is a rabbit and it hops. (Alternatively, some rabbits hop. Alternatively some hopping animals are rabbits.)

Problem 16 Determine the truth value of each of these statements if the universe of discourse of each variable consists of all real numbers

(a) $\exists x(x^2 = 2)$

(b) $\exists x(x^2 = -1)$

(c) $\forall x(x^2 + 2 \geq 1)$

(d) $\forall x(x^2 \neq x)$

Solution

(a) True

(b) False

(c) True

(d) False

Problem 20 Suppose that the universe of discourse of the propositional function $P(x)$ consists of -5,5,-3,-1,1,3, and 5. Express these statements without using quantifiers, instead using only negations, disjunctions, and conjunctions.

(a) $\exists x P(x)$

(b) $\forall x P(x)$

(c) $\forall x((x \neq 1) \rightarrow P(x))$

(d) $\exists x((x \geq 0) \wedge P(x))$

(e) $\exists x(\neg P(x)) \wedge \forall x((x \leq 0) \rightarrow P(x))$

Solution

(a) $P(-5) \vee P(-3) \vee P(-1) \vee P(1) \vee P(3) \vee P(5)$

(b) $P(-5) \wedge P(-3) \wedge P(-1) \wedge P(1) \wedge P(3) \wedge P(5)$

(c) $P(-5) \wedge P(-3) \wedge P(-1) \wedge P(3) \wedge P(5)$

(d) $P(1) \vee P(3) \vee P(5)$

(e) $(\neg P(-5) \vee \neg P(-3) \vee \neg P(-1) \vee \neg P(1) \vee \neg P(3) \vee \neg P(5)) \wedge (P(-5) \wedge P(-3) \wedge P(-1))$
Simplified as: $\neg(P(1) \wedge P(3) \wedge P(5)) \wedge (P(-5) \wedge P(-3) \wedge P(-1))$

Problem 32 Express each of these statements using quantifiers. Then form the negation of the statement so that no negation is to the left of a quantifier. Next, express the negation in simple English. (Do not simply use the words "It is not the case that")

(c) Every koala can climb

(d) No monkey can speak French

(e) There exists a pig that can swim and catch fish

Solution

(c) Let the universe is all koalas. $Q(x)$ denotes "x can climb", then "Every koala can climb" can be represented as $\forall x Q(x)$. The negation of it is $\exists x \neg Q(x)$. In English, it is "There exists a koala which cannot climb."

(d) Let the universe is all monkeys. $Q(x)$ denotes "x can speak French", then "no monkey can speak French" can be represented as $\forall x \neg Q(x)$. The negation is $\exists x Q(x)$. In English, it is "There exists a monkey which can speak French."

(e) Let the universe is all pigs. $P(x)$ and $Q(x)$ denote, "x can swim" and "x can catch fish" respectively. Then the sentence can be represented as $\exists x(P(x) \wedge Q(x))$. The negation is $\forall x(\neg P(x) \wedge \neg Q(x))$. In English, " A pig can not either swim or catch fish."

Problem 50 Show that $\forall x P(x) \vee \forall x Q(x)$ and $\forall x(P(x) \vee Q(x))$ are NOT logically equivalent.

Solution

$\forall x P(x) \vee \forall x Q(x)$ means $P(x)$ is true for every x OR $Q(x)$ is true for every x.

$\forall x(P(x) \vee Q(x))$ means for every x, either $P(x)$ or $Q(x)$ is true. For example, "for all 30 students in a class, 25 of them have only visited Mexico and the other 5 have only visited Canada". Let $P(x)$ represent "x visited Mexico" and $Q(x)$ represents "x visited Canada." In this case, $\forall x P(x) \vee \forall x Q(x)$ is false but, $\forall x(P(x) \vee Q(x))$ is true.

2 Section 1.4

Problem 6 Let $C(x, y)$ mean that student x is enrolled in class y, where the universe of discourse for x consists of all students at USC and the universe of discourse for y is the set of all classes being given at USC. Express each of these statements by a simple English sentence

(a) C(Randy Goldberg, CS 252)

(b) $\exists x C(x$, Math 695)

(c) $\exists y C(\text{Carol Sitea}, y)$

(d) $\exists x(C(x,\text{Matt 222}) \wedge C(x, \text{CS 252}))$

(e) $\exists x \exists y \forall z((x \neq y) \wedge (C(x,z) \rightarrow C(y,z)))$

(f) $\exists x \exists y \forall z((x \neq y) \wedge (C(x,z) \leftrightarrow C(y,z)))$

Solution **(a)** Randy Goldberg is enrolled in class CS 252.

(b) At least one student in USC is enrolled in class Math 695

(c) Carol Sitea is enrolled in at least one class given by USC

(d) There exists a student in USC who is enrolled in both Math 222 and CS 252

(e) There exists two USC students x and y, y is enrolled in all the classes x is enrolled

(f) There exists two USC students x and y, they are enrolled in exactly the same classes

Problem 10 Let $F(x,y)$ be the statement "x can fool y", where the universe of discourse consists of all the people in the world. Use quantifiers to express each of these statements.

(a) Everybody can fool Fred.

(b) Everybody can fool everybody.

(c) Everybody can fool somebody.

(d) There is no one who can fool everybody.

(f) No one can fool both Fred and Jerry.

(g) Nancy can fool exactly two people.

(h) There is exactly one person whom everybody can fool.

(i) No one can fool himself or herself.

(j) There is someone who can fool exactly one person besides himself.

Solution **(a)** $\forall x F(x, Fred)$

(b) $\forall x \forall y F(x,y)$

(c) $\forall x \exists y F(x,y)$

(d) $\neg(\exists x \forall y F(x,y))$ i.e. $\forall x \exists y \neg F(x,y)$

(f) $\forall x(\neg(F(x, Fred) \wedge F(x, Jerry)))$

(g) $\exists x \exists y \forall z(F(Nancy, x) \wedge F(Nancy, y) \wedge (x \neq y) \wedge (((z \neq y) \wedge (z \neq x)) \rightarrow \neg F(Nancy, z)))$

(h) $\exists x \forall y \forall z \exists r(F(y,x) \wedge ((x \neq z) \rightarrow \neg F(r,z)))$

(i) $\forall x \neg F(x,x)$

(j) $\exists x \exists y \forall z(F(x,y) \wedge (x \neq y) \wedge (((z \neq x)) \rightarrow \neg F(x,z))))$

Problem 22 Use predicates, quantifiers, logical connectives, and mathematical operators to expres the statement that there is a positive integer that is not the sum of three squares.

Solution Let the universe of discourse consist of all positive integers. Then:
$\exists p \forall x \forall y \forall z(p \neq x^2 + y^2 + z^2)$

Problem 40 Find a counterexample, if possible, to these universally quantified statements, where the universe of discourse for all variables consists of all integers

(b) $\forall x \exists y(y^2 - x < 100)$

Solution Counterexample: Let $x = -101$, we cannot find an integer y such that $y^2 < -1$.

3 Section 1.5

Problem 5 Use rules of inference to show that hypotheses, " Randy works hard,""If Randy works hard, then he is a dull boy," and "If Randy is a dull boy, then he will not get the job" imply the conclusion "Randy will not get the job."

Solution Let w be the proposition "Randy works hard", let d be the proposition, "Randy is a dull boy," and let j be the proposition, "Randy will get the job." We are given premises w, $w \to d$, and $d \to \neg j$. We want to conclude $\neg j$. We set up the proof in two columns, with reasons:

Step	Reason
1. w	Hypothesis
2. $w \to d$	Hypothesis
3. d	Modus Ponens using (1) and (2)
4. $d \to \neg j$	Hypothsis
5. $\neg j$	Modus Ponens using (3) and (4)

Problem 6 Construct an argument using rules of inference to show that the hypotheses, "If it does not rain or if it is not foggy, then the sailing race will be held and the life saving demonstration will go on, " "If the sailing race is held, then the trophy will be awareded, " and "The trophy was not awarded" imply the conclusion "It rained."

Solution Let r be the proposition, "It rains," let f be the proposition "It is foggy", let s be the proposition, "The sailing race will be held," let l be the proposition, "The life saving demonstration will go on", and let t be the proposition, "The trophy will be awarded." We are given premises $(\neg r \vee \neg f) \to (s \wedge l), s \to t$, and $\neg t$. We want to conclude r. We set up the proof in two columns, with reasons, as in Example 6. Note that it is valid to replace subexpressions by other expressions logically equivalent to them.

Step	Reason
1.$\neg t$	Hypothesis
2.$s \to t$	Hypothesis
3.$\neg s$	Modus tollens using Steps 1 and 2
4.$(\neg r \vee \neg f) \to (s \wedge l)$	Hypothesis
5.$(\neg(s \wedge l)) \to \neg(\neg r \vee \neg f)$	Contrapositive of Step 4
6.$(\neg s \vee \neg l) \to (r \wedge f)$	De Morgan's law and double negative
7.$(\neg s \vee \neg l)$	Addition, using Step 3
8.$(r \wedge f)$	Modus ponens using Steps 6 and 7
9.r	Simplification using Step 8

Problem 16 For each of these arguments determine whether the argument is correct of incorrect and explain why.

(a) Everyone enrolled in the university has lived in a dormitory. Mis has never lived in a dormitory. Therefore, Mia is not enrolled in the university.

(b) A convertible car is fun to drive. Isaac's car is not a convertible. Therefore, Isaac's car is not fun to drive.

Solution

(a) This is correct, using universal instantiation and modus tollens.

(b) this is not correct. After applying universal instantiation, it contains the fallacy of denying hypothesis.

4 Proofs

Problem 1 Do two proofs. One forward and one by contradiction:

1. $p \rightarrow q$
2. $r \rightarrow s$
3. $p \vee r$

Prove $q \vee s$

Forward proof

4. $\neg p \vee q$	1, equiv
5. $\neg r \vee s$	2, equiv
6. $q \vee r$	3,4 resolution
7. $q \vee s$	5,6 resolution
QED	

Proof by contradiction

4. Assume $\neg(q \vee s)$	
5. $\neg q \wedge \neg s$	4, deMorgan
6. $\neg q$	5, simplification
7. $\neg s$	5, simplification
8. $\neg p$	1,6 MT
9. $\neg r$	2,7 MT
10. $\neg p \wedge \neg r$	8,9 conjenction
11. $\neg(p \vee r)$	10, deMorgan,
Contradiction with 3. QED	

Problem 2 First order logic proof.

1. $\exists x R(x)$
2. $\neg \exists x (R(x) \wedge S(x))$

Prove: $\exists x \neg S(x)$

Proof

3. $\forall x \neg(R(x) \wedge S(x))$	2, negative
4. $R(a)$	1,EI
5. $\neg(R(a) \wedge S(a))$	3,4 UI
6. $\neg R(a) \vee \neg S(a)$	5, deMorgan
7. $\neg S(a)$	4,6 resolution
8. $\exists x \neg S(x)$	7, EG
QED	