Identification and Linear Estimation of General Dynamic Programming Discrete Choice Models

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Abstract

This paper studies the nonparametric identification and estimation of the structural parameters, including the per period utility functions, discount factors, and state transition laws, of general dynamic programming discrete choice (DPDC) models. I show an equivalence between the identification of general DPDC model and the identification of a linear GMM system. Using such an equivalence, I simplify both the identification analysis and the estimation practice of DPDC model. First, I prove a series of identification results for the DPDC model by using rank conditions. Previous identification results in the literature are based on normalizing the per period utility functions of one alternative. Such normalization could severely bias the estimates of counterfactual policy effects. I show that the structural parameters can be nonparametrically identified without the normalization. Second, I propose a closed form nonparametric estimator for the per period utility functions, the computation of which involves only least square estimation. The existing estimation procedures rely on assuming that the dynamic programming (DP) problem is stationary or on solving the DP problem numerically with the aid of terminal conditions. Neither the identification nor the estimation requires terminal conditions, the DPDC model to be stationary, or having a sample that covers the entire decision horizon.

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1 Introduction

The dynamic programming discrete choice (DPDC) model is an empirical framework for studying the intertemporal discrete choices in fields as labor economics and empirical industrial organization (see Ackerberg, Benkard, Berry, and Pakes 2007 and Keane, Todd, and Wolpin 2011 for surveys of applications). The DPDC model extends the static discrete choice model by allowing an individual’s current choice to affect not only her current utility but also her future state. Taking occupational choice as an example (Keane and Wolpin 1997), starting at some age, an individual can choose among different occupations/activities: attend schools, work in either a blue or white collar occupation, start her own business, work in military or stay at home (unemployed). Such an occupational choice can be made repeatedly through her lifetime. Her current occupational choice will affect not only her current utility but also her future human capital, and hence future income. For example, attending college is costly, but a college graduate is more likely to find a white collar job. It is reasonable to view her current occupational choice as a result of an intertemporal maximization of her expected lifetime utility. In DPDC model, this intertemporal optimization is modeled by the dynamic programming framework. The econometric problems in DPDC models are the identification and estimation of structural parameters, including the per period utility functions, discount factors, and state transition laws. If the structural parameters of a DPDC model are obtained, counterfactual policy interventions, such as the effect of subsidizing tuition on college enrollment, can be simulated.

The existing identification results and estimation methods of nonstationary DPDC models are both conceptually complicated and numerically difficult due to the complexity of nonstationary dynamic programming, that is a recursive solution method. This paper will show that the identification of nonstationary DPDC models and their estimation can be greatly simplified, because I will show that the identification of general DPDC models is equivalent to the identification of linear GMM system. So the identification of DPDC models can be understood from the familiar rank conditions in linear models. The per period utility functions and discount factors can be estimated by a closed form nonparametric linear estimator.

The idea is inspired by the econometric literature on dynamic game models. Pesendorfer and Schmidt-Dengler (2008), Bajari, Chernozhukov, Hong, and Nekipelov (2009), Bajari, Hong, and Nekipelov (2010) show that the Markovian equilibria of dynamic games with discrete choices can be equivalently written
as a system of equations linear in the per period utility functions. Hence the identification of per period utility functions in dynamic game models is similar to the identification of a linear GMM system. Moreover, the per period utility functions can then be estimated by the least squares. As a special case of the dynamic game with discrete choices, the identification and estimation of infinite horizon stationary single agent DPDC models can also be addressed using the equivalence to a linear GMM system [Pesendorfer and Schmidt-Dengler, 2008; Srisuma and Linton, 2012]. Because the equivalence to a linear GMM has greatly simplified our understanding of the identification of stationary DPDC models and their estimation, a natural question is if such an equivalence exists for general DPDC models, especially finite horizon nonstationary DPDC models. Finite horizon models are common in labor economics, since households live for a finite time. This paper addresses this question.

The DPDC model studied in this paper is general in three ways. First, the decision horizon can be finite or infinite. Second, all structural parameters, including per period utility functions, discount factors and transition laws, are allowed to be time varying. Third, I do not assume that the expected per period utility function associated with one particular alternative is known, or is normalized to be a known constant. This feature is important, because normalization of the period utility function will bias counterfactual policy predictions.

The normalization derives from the analogy between dynamic and static choice. In static discrete choice the conditional choice probabilities (CCP) only depend on the differences between the payoffs of alternatives. So we can change payoffs of alternatives so long as their differences are not changed. This ambiguity motivates the normalization of the payoff of one alternative [Magnac and Thesmar, 2002; Bajari, Benkard, and Levin, 2007; Pesendorfer and Schmidt-Dengler, 2008; Bajari, Chernozhukov, Hong, and Nekipelov, 2009; Blevins, 2014]. However, normalization in dynamic discrete choice models is not innocuous for counterfactual policy predictions. This point has been mentioned recently by some authors in a variety of settings, e.g. [Norets and Tang, 2014; Arcidiacono and Miller, 2014; Aguirregabiria and Suzuki, 2014; Kalouptsidi, Scott, and Souza-Rodrigues, 2015]. The intuition is that in a dynamic discrete choice model, a forward-looking individual’s current choice depends on future utility. This future utility depends on the per period utility functions of all alternatives. Consider the normalization of setting the per period utility of the first alternative to be zero for all states. Such a normalization will distort the

\[I\] also provide two propositions in the appendix showing the misleading consequence of normalization for counterfactual analysis.
effects of the current choice on future utility, because the per period utility of
the first alternative does not depend on the state. When we consider counter-
factual interventions, the effects of the current choice on counterfactual future
payoff will be also distorted, hence the counterfactual choice probability will be
biased.

Without imposing a normalization, I provide two alternative ways to iden-
tify the per period utility functions and discount factors. One is to assume
that there are excluded state variables that do not affect per period utilities
but affect state transitions. When excluded state variables are not available,
another way is to assume that per period utility function is time invariant but
that state transition laws hence continuation value functions are time varying.
The excluded variables restriction has been used to identify discount factors in
exponential discounting (Ching and Osborne, 2015) and hyperbolic discounting
(Fang and Wang, 2015). But it has not been used to identify per period util-
ity functions in general DPDC models. The closest work is Aguirregabiria and
Suzuki’s (2014) study of market entry and exit decisions, where the per utility
function is equal to the observable revenue net of unobservable cost. If firms’
dynamic programming problem is stationary, and the discount factor is known,
they use exclusion restrictions to identify the cost function. However they do
not consider the identification of the discount and nonstationary DPDC models.

Let us consider a binary choice model to explain the intuition why the exclusion
restrictions can identify the per period utility function without normalization.
The observable CCP is determined by the difference between the payoffs of the
two alternatives. In DPDC model, such a payoff difference is the sum of the
difference between per period utility functions, and the difference between the
discounted continuation value functions. The above mentioned two restrictions
create “exogenous” variation that can identify the value functions from the CCP.
The identification of the per period utility functions follows from the Bellman
equation.

Using the equivalence to a linear GMM, the estimation of DPDC models
becomes so simple that the per period utility functions and discount factors can
be estimated by a closed form nonparametric linear estimator after estimating
the conditional choice probabilities (CCP) and the state transition distributions.
The linear estimation procedure is hardly a surprise, because the structural pa-
rameters of DPDC models (including per period utility functions) are identified
from a linear system. The implementation of our linear estimator is simple be-
cause only basic matrix operations are involved. Moreover, our linear estimator
can be applied to situations where the agent’s dynamic programming problem
is nonstationary, the panel data do not cover the whole decision period, and there are no terminal conditions available. Such a simplicity in computation and flexibility in modeling are desirable in practice, because the existing estimation algorithms (Rust, 1987; Hotz and Miller, 1993; Aguirregabiria and Mira, 2002; Su and Judd, 2012) depend on complicated numerical optimization and/or iterative updating algorithms, and many of them cannot be applied when the dynamic programming problem is nonstationary and no terminal conditions are available.

1.1 Literature review

We now survey the literature. If the agent’s dynamic programming problem is stationary\textsuperscript{2} Rust (1994, section 3.5) shows that the structural parameters of the DPDC models, including the per period utility functions and the discount factor, are nonparametrically unidentified. However the exact degree of underidentification is not clear. Magnac and Thesmar (2002) extend Rust’s underidentification argument in two ways. First, they determine the exact degree of underidentification in a two periods DPDC model and discuss the identifying power of various restrictions (section 2 to 4 of their paper). Their conclusion is that the alternative specific per period utility functions cannot be nonparametrically identified if the distribution of the unobserved payoff shocks, the discount factor, and the per period utility function and the alternative specific value function (ASVF) associated with one specific alternative are not all known. The precise definition of the ASVF will be given later; at this moment, one just needs to understand that the ASVF is the best expected remaining lifetime payoff if that particular alternative is chosen. Second, they study the identification of DPDC models with unobserved heterogeneity (section 5). The unobserved heterogeneity in their paper is discrete and affects per period utility functions but does not affect the law of state transitions. Their conclusion is that the DPDC models with unobservable heterogeneity are nonparametrically unidentified even under strong restrictions, such as that the current and future payoffs of one alternative are assumed to be known.

The identification and estimation of the stationary single agent DPDC model is closely related to the identification of dynamic game models with Markov perfect equilibria. The crucial observation is that if dynamic programming problem is stationary, the Bellman equation becomes of a Fredholm integral

\textsuperscript{2}The agent’s dynamic programming problem is stationary if the per period utility functions, the law of state transitions and the discount factor of future payoff are all time invariant, and the decision horizon is infinite.
equation of type 2 from which the value function has to be solved. This implies that we have a closed form representation of the value function in terms of the per period utility function, discount factor and the CCP. Because the CCP can be written as a function of per period utility function, discount factor and the value function, there is an equation for the per period utility functions and discount factor by substituting the value function in the CCP with the close form representation of the value function. Pesendorfer and Schmidt-Dengler (2008), Bajari, Chernozhukov, Hong, and Nekipelov (2009), Bajari, Hong, and Nekipelov (2010) used this equation to study the identification of dynamic game models. Srisuma and Linton (2012) studied the identification and estimation of stationary single agent DPDC model when some of the state variables are continuous.

Blevins (2014) studies the nonparametric identification of the stationary dynamic programming decision process when the decisions involve both discrete and continuous choice. In the first stage, an agent makes a discrete choice; in the second stage, the agent makes a continuous choice given her previous discrete choice. Such a two-stage specification was also used in Bajari, Benkard, and Levin (2007). Bajari, Benkard, and Levin focus on the estimation issues, and their analysis allows for dynamic games. The advantage of using such a two-stage specification is that once the policy function of continuous choice is identified, the optimal continuous choice can be viewed as an observable state variable. Thus, Blevins model becomes a stationary DPDC model. Blevins conclusion is that when the discount factor, the per period utility function of one specific alternative, and the distribution of preference shocks are known, the per period utility functions of the other alternatives are nonparametrically identifiable. This conclusion corresponds to the earlier observation in Magnac and Thesmar (2002). When the distribution of preference shocks is unknown, he provided some exclusion restrictions (Assumption 12 on p. 546 of his paper) that can lead to identification of the distribution of the differences between payoff shocks. His method for identifying the distribution is similar to the control function approach used in the nonparametric instrumental variable literature (e.g., Blundell and Powell, 2004; Imbens and Newey, 2009). His identification arguments depend crucially on the stationarity assumption, without which the functional mapping in his proof does not exist. When the distribution of preference shocks is unknown, Norets and Tang (2014) provide a partial identification approach to analyze the stationary DPDC models when all observable state variables are discrete with finite support. Heckman and Navarro (2007) and Aguirregabiria (2010) study the identi-
fication of nonstationary DPDC models with a finite decision horizon. Both papers assume that researchers can observe the “outcomes” of agents’ choices. For example, the outcome is one’s earnings in Heckman and Navarro’s schooling decisions study. An agent’s per period utility is assumed to be the outcome net of the unobservable cost of choice, and hence the identification of utility function is then equivalent to the identification of the cost function. Heckman and Navarro identify the period utility function under several restrictions. The most substantial two restrictions are (1) the continuation value associated with one specific alternative is known, and (2) the transition between the observed states does not depend on the agent’s decisions. These assumptions are restrictive in practice.

Without these assumptions above, Aguirregabiria aims to identify the effects of certain counterfactual policy interventions rather than the structural parameters when the policy effects on the agents’ per period utility functions are completely known. There are two limitations of his approach. First, his method applies only to counterfactual policy interventions that affect the per period utility functions and the effects of which on current utility are completely known. If the intervention effects are unknown or the interventions are on state transitions, his method cannot be applied. Second, identification and estimation statements are based on backward induction, and the estimation requires data about decisions in the last period. It is not clear that if this method can be extended to deal with infinite horizon DPDC with short panel data.

The estimation of a DPDC model is usually complicated since the model is based on dynamic programming that is a recursive solution method. Researchers usually adopt the maximum likelihood method to estimate the structural parameters, although the identification in a maximum likelihood framework is unknown either. The first estimation method was the nested fixed-point (NFXP) algorithm proposed by Rust (1987). To alleviate computational burden, Hotz and Miller (1993) developed a semiparametric two-step estimator of the structural parameters. The first step is to estimate the CCP nonparametrically. The second step uses the famous Hotz and Miller inversion proposition that gives a representation of the ASVF in terms of the CCP, per period utility functions, and the discount factor. Consequently, one has a closed form representation of the CCP in terms of the CCP itself and the structural parameters (see equation (3.12) of Hotz and Miller’s paper). Substituting nonparametric estimates ob-

\footnote{In the working paper version (Aguirregabiria 2005), he did study the identification and estimation when decision horizon is infinite. But there he has to assume that the dynamic programming problem is stationary, and the estimation procedure becomes computationally difficulty because some contraction mappings are involved in his procedure.}
tained in the first step for the CCP in the closed form representation, one has the CCP for each value of structural parameters. Equating these expressions with its nonparametric estimates, one can develop a GMM estimator of the structural parameters, and this is the second step of Hotz and Miller’s estimation method. Also, Hotz and Miller’s idea also holds for nonstationary DPDC models. There are two potential limitations of the Hotz and Miller two-step estimator. First, the computational gain comes at the expense of both finite and asymptotic efficiency. This drawback has been addressed by Aguirregabiria and Mira (2002). Second, the closed form representation of the ASVF becomes complicated when there are many future periods left before the end of the decision horizon. The complication comes from the fact that the representation of the ASVF, see equation (3.12) of Hotz and Miller’s paper, requires the evaluation of the probabilities of all possible future paths and the expected utilities associated with these paths. This has not been noticed in the literature because the existing estimators focus on the stationary DPDC models, and stationarity provides a better way of expressing the ASVF in terms of the CCP and structural parameters (see equation (8) of Aguirregabiria and Mira, 2002, for example). Aguirregabiria and Mira (2002) provide a new approach called the nested pseudo likelihood (NPL) algorithm to estimate stationary DPDC models when the state variables are discrete. Their estimator could be as efficient as Rust’s NFXP estimator but computationally easier. When the dynamic programming process is stationary, Aguirregabiria and Mira establish a contraction mapping between the CCP. Using this contraction mapping, Aguirregabiria and Mira’s NPL estimator can improve the estimate of the CCP used in the second step of Hotz and Miller’s two-step estimator. Recently, Su and Judd (2012) provide another estimation approach called the mathematical program with equilibrium constraints (MPEC) for the stationary DPDC model. The equilibrium constraint in stationary DPDC models corresponds to the integrated Bellman equation. Their idea is to treat the ex ante value function, which becomes a vector when the observable states are discrete, as a parameter in maximizing the log likelihood function subject to the constraint that the ex ante value function must solve the integrated Bellman equation. However, their method works only with discrete state variables, and the number of points in the support has to be small. It is also not clear whether their method can be used to estimate nonstationary DPDC models.
1.2 Structure of the paper and notation rules

In section 2, we develop the dynamic programming discrete choice model of which identification and estimation will be studied. The model’s set up follows the literature, except that we allow per period utility functions and discount factors to be time varying. In section 3, we show that the identification of the DPDC model is equivalent to the identification of a linear GMM system, and provide a list of identification results under various restrictions. In particular, we show two ways to identify the DPDC models without normalizing per period utility functions. After clarifying the identification of the model, we show that the DPDC model can be estimated by simple linear estimators. Numerical experiments are conducted to check the performance and highlight some issues with our estimator. The last section concludes the paper with a discussion of some extensions of this paper.

Notation rules: Let $X$, $Y$ and $Z$ be three random variables. We write $X \perp \perp Y$ to denote that $X$ and $Y$ are independent. And write $X \perp \perp Y|Z$ to denote that $X$ and $Y$ are independent conditional on $Z$. Suppose random element $X$ can take only a finite number of values, e.g. the support of $X$ is $\mathcal{X} \equiv (x_1, \ldots, x_{d_x})$. Let $f : \mathcal{X} \mapsto \mathbb{R}$ be a real function. We use $\tilde{f}$ to denote the $d_x$-dimensional vector $(f(x_1), \ldots, f(x_{d_x}))^T$. For a real number $a \in \mathbb{R}$, let $a_n \equiv (a, \ldots, a)^T$ be an $n$-dimensional vector, whose entries are all $a$.

2 Dynamic Programming Discrete Choice Model

We first set up the dynamic programming discrete choice model. A female labor force participation example then follows the abstract setup to make the notations concrete.

We restrict our attention to binary choice case. The extension to multinomial choices is straightforward at the expense of more notations (see remark in section 3). In each period $t = 1, \ldots, T_* \leq \infty$, an agent makes a sequence of binary choices $\{Y_r \in \{0, 1\} : r = t, \ldots, T_* \leq \infty\}$ to maximize the expected life-time utility

$$u_i^{Y_t}(S_t, \varepsilon_i^{Y_t}) + \sum_{r=t+1}^{T_*} \left\{ \prod_{j=t}^{r-1} \delta_j \right\} \mathbb{E}_{r} \left\{ u_r^{Y_r}(S_r, \varepsilon_r^{Y_r}) \right| S_t, \varepsilon_i^{Y_t}, Y_t \right\}.$$  

For each period $t$, (i) the vector of state variables is $(S_t, \varepsilon_t^0, \varepsilon_t^1)$, where $\varepsilon_t^0$ and $\varepsilon_t^1$ are utility shocks associated with choices 0 and 1, respectively; (ii) $u_i^{Y_t}$ is the period utility function associated with choice $Y_t$, depending on state $(S_t, \varepsilon_i^{Y_t})$;
(iii) $\delta_t$ is the discount factor. The agent observes all states $(S_t, \varepsilon_t^0, \varepsilon_t^1)$, but researchers can only observe $S_t$. Note that both period utility functions and discount factors are allowed to be time varying. The following assumptions will be maintained throughout the paper.

**Assumption 1.**

(i) The observable state $S_t$ is discrete with finite time-invariant support $S \equiv \{s_1, \ldots, s_d\}$.

(ii) Let $\varepsilon_t \equiv (\varepsilon_t^0, \varepsilon_t^1)^\top$. The sequence of utility shocks $\{\varepsilon_t : t = 1, \ldots, T\}$ is independent and identically distributed (iid) with known CDF $G$. \[ P(\varepsilon_t \mid S_t, Y_t) = G(\varepsilon_t). \]

(iii) Let $\tilde{G}$ be the CDF of $\varepsilon_t^0 - \varepsilon_t^1$. Assume $CDF \tilde{G}$ is continuous and strictly increasing.

(iv) For each period $t$, the utility shocks $\varepsilon_t \perp \{S_t, \{S_r, Y_r : r = 1, \ldots, t-1\}\}$. \[ \mathbb{P}(\varepsilon_t \mid S_t, Y_t) = \mathbb{P}(\varepsilon_t) = \tilde{G}(\varepsilon_t). \]

(v) For each period $t$, $S_{t+1} \perp \{\varepsilon_t, \{S_r, \varepsilon_r, Y_r : r = 1, \ldots, t-1\}\} | (S_t, Y_t)$. \[ \mathbb{P}(S_{t+1} \mid S_t, Y_t) = \mathbb{P}(S_{t+1} \mid S_t) G(\varepsilon_t), \]

**Assumption 2.** For each alternative $d \in \{0, 1\}$ and period $t$, the period utility function $u_t^d(S_t, \varepsilon_t^d)$ is additive in utility shock $\varepsilon_t^d$. In particular, let $u_t^d(S_t, \varepsilon_t^d) \equiv \mu_t^d(S_t) + \varepsilon_t^d$.

By assumption the structural period utility function $\mu_t^d(S_t)$ in assumption 2 is finitely dimensional. Denote $\mu_t^d = (\mu_t^d(s_1), \ldots, \mu_t^d(s_d))^\top$. Moreover, assumption 1 implies that the observable state $S_t$ is a controlled first-order Markov chain. We have

$$
\mathbb{P}(S_{t+1}, \varepsilon_{t+1} \leq \varepsilon | S_t, \varepsilon_t, Y_t, \ldots, S_1, \varepsilon_1, Y_1) = \mathbb{P}(S_{t+1}, \varepsilon_{t+1} \leq \varepsilon | S_t, \varepsilon_t, Y_t)
= \mathbb{P}(S_{t+1} | S_t, Y_t) \mathbb{P}(\varepsilon_{t+1} \leq \varepsilon)
= f_{t+1}(S_{t+1} | S_t, Y_t) G(\varepsilon),
$$

for each $t$. Here $f_{t+1}(S_{t+1} | S_t, Y_t)$ is the conditional probability function of $S_{t+1}$ given $S_t$ and $Y_t$. Let $F_{t+1}^d$ be the state transition matrix describing the transition from state $S_t$ to $S_{t+1}$ when choice $Y_t = d \in \{0, 1\}$:

$$
F_{t+1}^d = \begin{bmatrix}
    f_{t+1}(s_1 | s_1, Y_t = d) & \ldots & f_{t+1}(s_d | s_1, Y_t = d) \\
    \vdots & \ddots & \vdots \\
    f_{t+1}(s_1 | s_d, Y_t = d) & \ldots & f_{t+1}(s_1 | s_d, Y_t = d)
\end{bmatrix}.
$$

These assumptions are all standard in the literature. We emphasize three important limitations implied by them. First, the condition that the unobservable
where \( \delta \) is also finitely dimensional by assumption 1.i. Let for any function \( v_t(S_t, \varepsilon_t) \)
\begin{equation}
(2.2) \quad v_t^d(S_t) \equiv \mu_t^d(S_t) + \delta_t \mathbb{E}^d_{t+1} \{ v_{t+1}(S_{t+1}) | S_t \}.
\end{equation}

Kristensen, Nesheim, and de Paula (2014) studies the two-step CCP estimation of dynamic
discrete choice models when the period utility function is not additive in utility shocks.
In terms of ASVF, the Bellman equation is

\[(2.3) \quad V_t(S_t, \varepsilon_t) = \max \{v_0^a(S_t) + \varepsilon_t^0, v_1^a(S_t) + \varepsilon_t^1\}.\]

In each period \(t\), the agent makes choice \(Y_t\) satisfying the Bellman equation, that is

\[(2.4) \quad Y_t = \arg \max_{d \in \{0, 1\}} v_d^a(S_t) + \varepsilon_t^d.\]

The above display implies that the ASVF \(v_d^a(S_t)\) is the expected payoff an agent considers in a dynamic discrete choice problem. By the decision making formula \((2.4)\), the CCP \(p_t(S_t) \equiv \mathbb{P}(Y_t = 1|S_t)\) is as follows,

\[(2.5) \quad p_t(S_t) = \mathbb{P}\{\varepsilon_t^0 - \varepsilon_t^1 < v_1^a(S_t) - v_0^a(S_t)\} = \tilde{G}\{v_1^a(S_t) - v_0^a(S_t)\}.\]

Recall \(\tilde{G}\) is the CDF of utility shocks difference \(\varepsilon_t^0 - \varepsilon_t^1\). Note that the CCP function \(p_t(S_t)\) is also finitely dimensional, and let \(p_t \equiv (p_t(s_1), \ldots, p_t(s_d))^T\).

**Example** (Female labor force participation model). Our particular model is based on Keane, Todd, and Wolpin (2011, section 3.1). In each year \(t\), a married woman makes a labor force participation decision \(Y_t \in \{0, 1\}\), where 1 is “to work” and 0 is “not to work”, to maximize the expected life-time utility.

The period utility depends on the household consumption \((\text{cons}_t)\) and the number of young children \((\text{kid}_t)\) in the household\(^5\). Consumption equals to the household’s income net of child-care expenditures. The household income is the sum of the husband’s income \((\text{husb}_t)\) and the wife wage \((\text{wage}_t)\) if she works. The per-child child-care cost is \(\beta\) if she works, and zero if she stays at home. So consumption is

\[\text{cons}_t = \text{husb}_t + \text{wage}_t \times Y_t - \beta \text{kid}_t \times Y_t.\]

Suppose the wage offer function takes the following form

\[\text{wage}_t = \alpha_1 + \alpha_2 xp_t + \alpha_3 (xp_t)^2 + \alpha_4 \text{edu} + \omega_t,\]

where \(xp_t\) is the working experience (measured by the number of prior periods the woman has worked) of the woman in year \(t\), \(\text{edu}\) is her education level, \(\omega_t\) is random shock, which is independent of the wife’s working experience and edu-

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\(^5\)We are not going to model the fertility decision here, and assume the arrival of children as an exogenous stochastic process.
cation. Assume the period utility functions associated with the two alternatives are

\begin{equation}
\begin{align*}
    u_1^t(S_t, \epsilon_1^t) &= \mu_1^t(husb_t, xp_t, edu_t, kid_t) + \epsilon_1^t \\
    &= husb_t + \alpha_1 + \alpha_2 xp_t + \alpha_3 (xp_t)^2 + \alpha_4 edu_t - \beta kid_t + \epsilon_1^t, \\
    u_0^t(S_t, \epsilon_0^t) &= \mu_0^t(husb_t, kid_t) + \epsilon_0^t.
\end{align*}
\end{equation}

Besides the observable state variables about the woman, we also observe her husband’s working experience \( xp^H_t \) and education level \( edu^H_t \). Given husband’s income \( husb_t \), these two state variables, \( xp^H_t \) and \( edu^H_t \), do not affect the period utility but affect the state transitions, especially by affecting husband’s future income. These two state variables excluded from the period utility function will be useful for identification of the structural parameters. Let \( S_t \equiv (husb_t, xp_t, edu_t, kid_t, xp^H_t, edu^H_t) \) be the vector of observable state variables.

The problem is dynamic because the woman’s current working decision \( Y_t \) affects her working experience in the next period: \( xp_{t+1} = xp_t + Y_t \). As in the general model, the woman’s choice \( Y_t \) maximizes the value function

\[ Y_t = \arg \max_{d \in \{0, 1\}} v^d_t(S_t) + \epsilon_t^d, \]

where the ASVF \( v^d_t(S_t) \) is defined by equation (2.2) with the period utility functions being substituted by equation (2.6).

We would be interested in predicting the labor supply effects of some counterfactual policy interventions, such as child-care subsidy, tax reduction or the introduction of contraceptive techniques to households. In terms of the CCP, this means we would like to know the new CCP after imposing these counterfactual policy interventions. To answer these questions, we first need to identify and estimate the structural parameters.

### 3 Identification of Structural Parameters

We start by clarifying the data as well as the structural parameters of our DPDC model. Assume that researchers only observe a subsequence of an agent’s dynamic decision process, rather than the whole process from the initial period \( 1^* \) to the decision horizon \( T^* \). Denote \( 1, 2, \ldots, T \) the sampling periods. For expositional simplicity, assume that the sampling periods \( 1, 2, \ldots, T \) are the first \( T \) periods in the decision horizon \( \{1^*, 2^*, \ldots, T^*\} \) with \( 1 \leq T \leq T^* \). So data in

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\(^6\)Our theory can accommodate the situation where the first sampling period 1 does not have to be the initial decision period \( 1^* \), neither does the last sampling period \( T \) have to be
our problem are $D \equiv (Y_1, \ldots, Y_T, S_1, \ldots, S_T)^T$ with support $D = \{0, 1\}^T \times S^T$ containing $2^T \times d^T$ values. Let $\theta$ denote the vector of structural parameters of this model including period utility functions $(\mu_0^t, \mu_1^t)$, discount factor $(\delta_t)$ and transition densities $(f_t)$ in each period $t$. It will be useful to reparameterize $(\mu_0^t, \mu_1^t)$ as $(\mu_1^t/0^t, \mu_0^t)$ for each $t$, where $\mu_1^t/0^t(S_t) = \mu_1^t(S_t) - \mu_0^t(S_t)$. Let $\theta_t \equiv (\mu_0^t, \mu_1^t/0^t, \delta_t, F_0^t, F_1^t)$ for $t = 1, \ldots, T_\ast$, or equivalently $\theta_t \equiv (\mu_0^t, \mu_1^t/0^t, \delta_t, f_0^t, f_1^t)$. So $\theta = (\theta_1, \ldots, \theta_{T_\ast})$. Let $\Theta$ be the parameter space.

We consider identification in such a short panel data situation not only because short panel data are common in empirical studies, but also because the length of panel data turns out to play an important role in the identification of DPDC models. Simply put, one would need at least three consecutive periods to identify DPDC models without normalizing period utility functions. It is remarkable that such dependence of identification of DPDC models on the length of panel data has not been noticed in the current literature.\footnote{In the literature of the identification of the CCP with unobservable types, the identification also depends on the length of panel data. Interestingly, $T \geq 3$ is also required to identify type specific CCP (e.g. \cite{Kasahara and Shimotsu 2009, Hu and Shum 2012, Bonhomme, Jochmans, and Robin 2013, 2014}).}

Below, we first show that the identification of DPDC model is equivalent to the identification of linear GMM model. Then, applying this equivalence, we prove a list of identification results. Several important remarks are added at the end of this section.

### 3.1 Linear GMM representation of DPDC models

Recall the definition of identification in general. A structural model maps its primitive parameters $\theta \in \Theta$ to a joint probability function $f(D; \theta)$ of data $D \in D$. Let $\Gamma(\theta; D) : \theta \in \Theta \mapsto f(D; \theta) \in \mathcal{F} \equiv \{\Gamma(\vartheta; D) : \vartheta \in \Theta\}$ be one of such mappings embodied in the model, so $f(D; \theta) = \Gamma(\theta; D)$. Both the mapping $\Gamma(\cdot; D)$ and the parameter space $\Theta$ are determined by the model’s assumptions. Two parameters $\theta$ and $\tilde{\theta}$ are observationally equivalent if and only if (iff) $f(D; \theta) = f(D; \tilde{\theta})$ for all $D \in D$. Given data $D$, a set of structural parameters $\theta$ is identified in the parameter space $\Theta$ iff any two observationally equivalent parameters are identical. In terms of the mapping $\Gamma(\theta; D)$, the set of parameters $\theta$ is identified in $\Theta$ iff $\Gamma(\theta; D)$ is injective in $\Theta$. Viewing $f(D)$ as known and $\theta$ as unknown, the mapping $\Gamma(\theta; D)$ is injective in $\Theta$ is equivalent

the decision horizon $T_\ast$.\footnote{In the literature of the identification of the CCP with unobservable types, the identification also depends on the length of panel data. Interestingly, $T \geq 3$ is also required to identify type specific CCP (e.g. \cite{Kasahara and Shimotsu 2009, Hu and Shum 2012, Bonhomme, Jochmans, and Robin 2013, 2014}).}
to the condition that the system of equations

\[(3.1)\quad f(D) = \Gamma(\theta; D) : \text{for all } D \in D\]

subject to \(\theta \in \Theta,\)

has a unique solution for \(\theta\) with any \(f(D) \in \mathcal{F}\). When the above equation does not have a unique solution, the solution set of equation \(3.1\) is the identified set of \(\theta\). Due to the limited access to data and/or weak restrictions on the parameter space, we sometimes can only identify some components of the structural parameters, which turns out to be our case. Let \(\theta \equiv (\theta_a, \theta_b) \in \Theta\) be the vector of parameters. We say \(\theta_a\) is identified in \(\Theta\) iff for any pair \(\theta \equiv (\theta_a, \theta_b), \tilde{\theta} \equiv (\tilde{\theta}_a, \tilde{\theta}_b) \in \Theta,\) the condition that \(\theta\) and \(\tilde{\theta}\) are observationally equivalent implies \(\theta_a = \tilde{\theta}_a\). Again, this statement can be rephrased in terms of equation \(3.1\) as follows: \(\theta_a\) is identified in \(\Theta\) iff equation \(3.1\) has a unique solution of \(\theta_a\).

Applying the above general identification criterion requires finding the corresponding equation \(3.1\) for our DPDC model. We first determine the mapping \(\Gamma(\theta; D)\) for our DPDC model. Such a mapping \(\Gamma(\theta; D)\) must satisfy all assumptions of the model, so must any joint probability function \(f(D) \in \mathcal{F} \equiv \{\Gamma(\theta; D) : \theta \in \Theta\}\). The model has two assumptions: (i) the observable state \(S_t\) is a controlled first order Markov chain by assumption [1] and (ii) the choice \(Y_t\) is conditionally independent of the history \((S_{1}, Y_{1}, \ldots, S_{t-1}, Y_{t-1})\) given the current state \(S_t\). Consequently, any joint probability function \(f(D) \in \mathcal{F}\) can be decomposed as the product

\[(3.2)\quad f(D) = f_1(S_1) \times \left\{ \prod_{i=1}^{t-1} \mathbb{P}(Y_i|S_i)f_{t+1}(S_{t+1}|S_t, Y_t) \right\} \times \mathbb{P}(Y_T|S_T),\]

where \(\mathbb{P}(Y_i|S_i) = \{p_i(S_i)\}^{Y_i}\{1 - p_i(S_i)\}^{1-Y_i}\). The mapping \(\Gamma(\theta; D)\) can be constructed by

\[(3.3)\quad \Gamma(\theta; D) = f_1(S_1; \theta_1) \times \left\{ \prod_{i=1}^{t-1} \mathbb{P}(Y_i|S_i; \theta)f_{t+1}(S_{t+1}|S_t, Y_t; \theta_{t+1}) \right\} \times \mathbb{P}(Y_T|S_T; \theta_T),\]

where \(\mathbb{P}(Y_i|S_i; \theta) = \{p_i(S_i; \theta)\}^{Y_i}\{1 - p_i(S_i; \theta)\}^{1-Y_i}\) and \(p_i(S_i; \theta) = \pi_i(\theta; D)\).

Here \(\pi_i(\theta; D) : \theta \mapsto p_i(S_i; \theta)\) is a mapping from parameters \(\theta\) to the CCP in period \(t\). We added \(\theta_1\) in \(f_1(S_1; \theta_1)\) and \(\theta_{t+1}\) in \(f_{t+1}(S_{t+1}|S_t, Y_t; \theta_{t+1})\) to emphasize that they are part of the structural parameters \(\theta\). To completely specify the mapping \(\Gamma(\theta; D)\), we then only need to find the sequence of CCP mapping \(\{\pi_i(\theta; D) : t = 1, \ldots, T\}\).

The attention now is to determine the form of CCP mapping \(\pi_i(\theta; D_t)\). First,
it follows from the definition of CCP in equation (2.5) that

\[ p_t(S_t) = \tilde{G}\left\{v_t^1(S_t) - v_t^0(S_t)\right\} \]
\[ = \tilde{G}\left[\mu_t^{1/0}(S_t) + \delta_t E_{t+1}^{1/0}\{v_{t+1}(S_{t+1})|S_t\}\right], \]

where \( \tilde{G} \) is the CDF of the utility shocks difference \( \varepsilon_0^t - \varepsilon_1^t \). The second line follows from plugging the definitions of ASVF \( v_t^d(S_t) \) in equation (2.2). Because CDF \( \tilde{G} \) is strictly increasing (assumption 1.iii), its inverse \( \tilde{G}^{-1} \) exists. Let \( \phi(\cdot) \equiv \tilde{G}^{-1}(\cdot) \) denote the inverse.

Second, it follows from the definition of ex ante value function \( v_t(S_t) \) in equation (2.1) and the Bellman equation (2.3) that

\[ v_t(S_t) \equiv \hat{V}_t(S_t, \varepsilon_t) dG(\varepsilon_t) \]
\[ = \int \max\{v_t^0(S_t) + \varepsilon_t^0, v_t^1(S_t) + \varepsilon_t^1\} dG(\varepsilon_t) \]
\[ = v_t^0(S_t) + \int \max\{\varepsilon_t^0, v_t^1(S_t) - v_t^0(S_t) + \varepsilon_t^1\} dG(\varepsilon_t) \]
\[ = v_t^0(S_t) + \int \max\{\varepsilon_t^0, \phi(p_t(S_t)) + \varepsilon_t^1\} dG(\varepsilon_t) \]
\[ = v_t^0(S_t) + \psi(p_t(S_t)) \]

where \( \psi \) depends only on the CDF \( G \) of the utility shocks \( \varepsilon_t \equiv (\varepsilon_t^0, \varepsilon_t^1)^\top \). If \( (\varepsilon_t^0, \varepsilon_t^1) \) follows independent standard type-1 extreme value distribution, \( \psi(p) = \gamma - \ln(1 - p) \), where \( \gamma \approx 0.5772 \) is the euler’s constant. Substituting \( v_t^0(S_t) \) in the above display with its definition, \( v_t^0(S_t) = \mu_t^0(S_t) + \delta_t E_{t+1}^0\{v_{t+1}(S_{t+1})|S_t\} \), we have a recursive expression of the ex ante value function,

\[ v_t(S_t) = \mu_t^0(S_t) + \delta_t E_{t+1}^0\{v_{t+1}(S_{t+1})|S_t\} + \psi(p_t(S_t)). \]

Define \( \theta_a \equiv \{\theta_1, \ldots, \theta_{T-1}\} \) and \( \theta_b \equiv \{\theta_T, \ldots, \theta_T\} \), so \( \theta = (\theta_a, \theta_b) \). The

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8For multinomial discrete choices model, such an invertibility still holds due to the well known Hotz-Miller inversion results (Proposition 1 of their paper [Hotz and Miller, 1993] or more explicitly Lemma 1, 2, and 3 in [Arcidiacono and Miller, 2011].
sequence of mapping \( \{ \pi_t(\theta; D) : t = 1, \ldots, T \} \) can be formed as follows,

\[
p_T(S_T) = \tilde{G}(v^1_T(S_T; \theta_0) - v^0_T(S_T; \theta_0)),
\]

\[
v_T(S_T) = v^0_T(S_T; \theta_0) + \psi(p_T(S_T)),
\]

\[
p_t(S_t) = \tilde{G}[\mu^0_t(S_t) + \delta_t \mathbb{E}_{t+1}^0 \{ v_{t+1}(S_{t+1}) | S_t \}],
\]

\[
v_t(S_t) = \mu^0_t(S_t) + \delta_t \mathbb{E}_{t+1}^0 \{ v_{t+1}(S_{t+1}) | S_t \} + \psi(p_t(S_t)), \quad t = 1, \ldots, T - 1.
\]

The structural parameters \( \theta_b \) appear explicitly in the above display, but parameters \( \theta_b \) appear only implicitly through the two ASVF \( v^0_T(S_T; \theta_b) \) and \( v^1_T(S_T; \theta_b) \) in the last sampling period \( T \). The ASVF \( v^0_T(S_T; \theta_b) \) and \( v^1_T(S_T; \theta_b) \) are uniquely determined by \( \theta_b \).

For our \( f(D) \) and \( \Gamma(\theta; D) \) in equation (3.2) and (3.3), respectively, the equation \( f(D) = \Gamma(\theta; D) \) in (3.1) is equivalent to the following\(^9\)

\[
(3.7a) \quad \phi(p_T(S_T)) = v^1_T(S_T; \theta_b) - v^0_T(S_T; \theta_b)
\]

\[
\phi(p_t(S_t)) = \mu^0_t(S_t) + \delta_t \mathbb{E}_{t+1}^0 \{ v_{t+1}(S_{t+1}) | S_t \}, \quad t = 1, \ldots, T - 1,
\]

\[
v_t(S_t) = \mu^0_t(S_t) + \delta_t \mathbb{E}_{t+1}^0 \{ v_{t+1}(S_{t+1}) | S_t \} + \psi(p_t(S_t)), \quad t = 1, \ldots, T - 1,
\]

\[
(3.7b) \quad v_T(S_T) = v^0_T(S_T; \theta_b) + \psi(p_T(S_T)),
\]

\[
(3.7c) \quad f_t(S_t | S_{t-1}, Y_{t-1}) = f_t(S_t | S_{t-1}, Y_{t-1} ; \theta_{t-1}), \quad t = 2, \ldots, T,
\]

\[
(3.7d) \quad f_1(S_1) = f_1(S_1 ; \theta_1),
\]

\( \forall S_t \in S, \) subject to \( \theta \in \Theta, \)

because \( \tilde{G}(\cdot) \) is invertible with inverse \( \phi(\cdot) \). So the identification of structural parameters in our DPDC models can be approached by checking if equation (3.7) has unique solutions for the parameters. According to this criterion, the state transition laws \( \{ f_1(S_1; \theta_1), f_t(S_t | S_{t-1}, Y_{t-1} ; \theta_{t-1}) : t = 2, \ldots, T \} \) are of course identified. So we can remove equations equation (3.7c) and (3.7d), and treat state transition laws as known. Another observation is that the identification of \( \theta_a \equiv (\theta_1, \ldots, \theta_{T-1}) \) depends on \( v_T(S_T) \) but not on \( v^0_T \) or \( v^1_T \). Equation (3.7a) and (3.7b) simply state that \( v_T \) can be reparameterized as \( v_T^0 \) or \( v_T^1 \). So we can ignore equation (3.7a) and (3.7b) in the identification of \( \theta_a \). The identification\(^9\)

\( \text{For example, if } T_a = T + 1, \) \( v^1_T(S_T) = \mu^1_a(S_T) + \delta_T \mathbb{E}_{T+1}^1 \{ v_{T+1}(S_{T+1}) | S_T \}, \) where \( v_{T+1}(S_{T+1}) = \mu^0_{T+1}(S_{T+1}) + \psi(p_{T+1}(S_{T+1}) \not= \tilde{G}(\mu^0_{T+1}(S_{T+1})). \)

\( \text{It is remarkable that we did not include equation (3.6) for the case } d = 1, \) because equation (3.5) equals to the difference between equation (3.6) evaluated for \( d = 1 \) and \( d = 0. \)
of \( \theta \) will be based on the following system of equations:

\[
\begin{align*}
\mu_t^{1/0}(S_t) + \delta_t \mathbb{E}_t^{1/0} \{v_{t+1}(S_{t+1})|S_t\} &= \phi[p_t(S_t)] \\
v_t(S_t) - \mu_t^{0}(S_t) - \delta_t \mathbb{E}_t^{0} \{v_{t+1}(S_{t+1})|S_t\} &= \psi[p_t(S_t)]
\end{align*}
\]

(ID)

for all \( S_t \in \mathcal{S} \) and \( t = 1, \ldots, T - 1 \), subject to \( \theta \in \Theta \).

In this system of equations, the \textit{known objects} are functions of CCP \( \{\phi[p_t(S_t)], \psi[p_t(S_t)] : t = 1, \ldots, T - 1\} \) and transition matrices \( \{F_2^d, \ldots, F_T^d : d = 0, 1\} \) hidden in the conditional expectation operators \( \mathbb{E}_t^{1/0}(\cdot|S_t) \) and \( \mathbb{E}_t^{0}(\cdot|S_t) \), and the \textit{unknowns} are period utility functions \( \{\mu_t^{0}, \mu_t^{1/0} : t = 1, \ldots, T - 1\} \), expected value functions \( \{v_t : t = 1, \ldots, T\} \), and the discount factors \( \{\delta_t : t = 1, \ldots, T - 1\} \). A component of the structural parameters \( \theta \), including \( \{\mu_t^{0}, \mu_t^{1/0}, \delta_t : t = 1, \ldots, T - 1\} \), is identified iff equation (ID) has a unique solution for it.

When the discount factors \( \delta_t \) are known, the uniqueness problem is very easy to address because equation (ID) is linear in all unknown parameters. More explicitly, using the notations of \( \mu_t^{0}, \mu_t^{1/0}, v_t \) and \( F_t^{0}, F_t^{1/0} \), equation (ID) can be written as follows,

\[
\begin{align*}
\mu_t^{1/0} + \delta_t F_t^{1/0} v_{t+1} &= \phi(p_t) \\
v_t - \mu_t^{0} - \delta_t F_t^{0} v_{t+1} &= \psi(p_t)
\end{align*}
\]

In this sense, the identification of DPDC models is equivalent to identification of a linear GMM system, henceforth a familiar problem. The necessary condition for identification is that the number of equations should be greater than the number of unknowns (order condition). If the order condition fails, we shall consider restrictions that can eliminate certain number of unknowns, or add more equations by increasing panel data length \( T \).

### 3.2 Identification of DPDC models by linear GMM representation

A sequence of identification results will be derived by using the linear GMM representation of DPDC in equation (ID). The unknowns in equation (ID) are \( \{\mu_t^{1/0}, \mu_t^{0}, v_t, \delta_t, v_T : t = 1, \ldots, T - 1\} \), or equivalently \( \{\mu_t^{1/0}, \mu_t^{0}, v_t, \delta_t, v_T : t = 1, \ldots, T - 1\} \). Without restriction, we then have \((3T - 2) \times d_s + (T - 1)\) unknowns but \((2T - 2) \times d_s + d_s\) equations in system (ID). This implies that the structural parameters are not identified even when all discount factors are known.

**Proposition 1** (Unidentification without restrictions). Suppose assumption [2] and [3] hold, and there are no additional restrictions. The state transition laws \( \{f_1(S_1), f_{t+1}(S_{t+1}|S_t, Y_t) : t = 1, \ldots, T - 1\} \) are identified, and no other struc-
atural parameters are identifiable.

The underidentification of DPDC models has long been aware in the literature (Rust [1994], Magnac and Thesmar [2002]). The problem of interests is what restrictions shall we use? We start by the following restrictions, which were first proposed by Magnac and Thesmar [2002] in a two period case.

**Restriction 1** (Normalization and known discounts). (i) The period utility function \( \mu_t^1 \) is known for each \( t = 1, \ldots, T - 1 \). The ASVF \( v_T^0(S_T) \) is known for every state \( S_T \in S \).

(ii) The discount factors \( \delta_1, \ldots, \delta_{T-1} \) are known.

**Proposition 2** (Identification with restriction 1). In addition to assumption 1 and 2, suppose restriction 1 also holds. Then the period utility functions \( \{\mu_1^1, \ldots, \mu_{T-1}^1\} \) are identified.

Restriction 1 works, because it removes exactly \( T \times d_s \) unknowns (there are \( T-1 \) discount factors, \( T-1 \) period utility functions \( \{\mu_1^0, \ldots, \mu_{T-1}^0\} \) and the ex ante value function \( v_T \)). We then have \( (2T-2) \times d_s \) unknowns and \( (2T-2) \times d_s \) equations that are linear in them, and the parameters are just-identified.

One simple yet important remark is that when \( T < T_s \) (short panel data), knowing the period utility functions \( \mu_1^0, \ldots, \mu_T^0 \) and discount factors \( \delta_1, \ldots, \delta_{T-1} \) is not enough for the identification of nonstationary DPDC models; we have to know \( v_T^0 \). The proof of Proposition 2 is trivial from our linear system equation (1D), hence omitted. For illustration purpose, let \( T = 2 \), which is the case in Magnac and Thesmar [2002] paper. First, we have \( v_2(S_2) = v_1^0(S_2) + \psi(p_2(S_2)) \) from the known \( v_2^0(S_2) \). Second, we have \( \mu_1^{1/0}(S_1) = \phi(p_1(S_1)) - \delta_1 E_{2}^{1/0}(\psi(S_2)) \), hence \( \mu_1^1(S_1) = \mu_1^{1/0}(S_1) + \mu_1^0(S_1) \).

In restriction 1, we assume the discount factors \( \delta_1, \ldots, \delta_{T-1} \) are known. As we said, this eliminates \( T - 1 \) unknowns. To identify discount factors, we need to find a different set of \( T - 1 \) restrictions as a substitute for the known discount factors restriction.

**Restriction 2** (Normalization and unknown discount). (i) Restriction 1

(ii) For each \( t = 1, \ldots, T - 1 \), there exist distinct \( s_t, s_t' \in S \) such that \( \mu_t^{1/0}(s_t) = \mu_t^{1/0}(s_t') \).

**Proposition 3** (Identification with restriction 2). In addition to assumption 1 and 2, suppose restriction 2 also holds. For period \( t = T - 1, T - 2, \ldots, 1 \), if \( E_{t+1}^{1/0}(\psi(S_{t+1})) \neq E_{t+1}^{1/0}(\psi(S_{t+1}')) \), the period utility \( \mu_t^1 \) and the discount factor \( \delta_t \) are identified. Here \( \psi_T(S_T) = v_T^0(S_T) + \psi(p_T(S_T)) \), and for
\[ t = T - 1, T - 2, \ldots, 1, v_t(S_t) = \mu_t^0(S_t) + \delta_t \mathbb{E}_{t+1}^0 \{ v_{t+1}(S_{t+1}) \mid S_t \} + \psi \{ p_t(S_t) \} \]

is constructed recursively.

Magnac and Thesmar (2002) also proposed an exclusion condition to identify discount factors. It is remarkable that their exclusion restriction is different from ours (see remark 3 at the end of this section).

It is usually not natural to assume that the ASVF \( v_0^T(S_T) \) is known as required in restriction 1 and 2. An alternative restriction one may consider is that the agent’s dynamic programming problem is stationary. This restriction was considered by many authors, e.g. Pesendorfer and Schmidt-Dengler (2008), Bajari, Chernozhukov, Hong, and Nekipelov (2009), Bajari, Hong, and Nekipelov (2010), Srisuma and Linton (2012), and Blevins (2014) among others. The identification with stationarity assumption can also be easily understood from the linear GMM equation (ID).

**Restriction 3** (Normalization and stationarity).  
(i) The agent’s dynamic programming problem is stationary, so that all structural parameters, including period utility functions and discount factors, are time invariant.

(ii) The discount factor \( \delta \) is known.

(iii) The period utility function \( \mu^0 \) is known.

**Proposition 4** (Identification with restriction 3). In addition to assumption 1 and 2, suppose restriction 3 also holds. Then the time invariant period utility function \( \mu^1 \) is identified.

Assuming stationarity can remove a large number of unknowns, but also a large number of equations. Under restriction 3 the linear system (ID) becomes

\[
\begin{align*}
\mu^{1/0}(S) + \delta \mathbb{E}^{1/0} \{ v(S') \mid S \} &= \phi \{ p(S) \} \\
v(S) - \mu^0(S) - \delta \mathbb{E}^0 \{ v(S') \mid S \} &= \psi \{ p(S) \}
\end{align*}
\]

which contains \( 2d_s \) unknowns (\( \mu^{1/0} \) and \( v \)) and \( 2d_s \) equations. The model is again just identified. It is remarkable that although restriction 1, 2 and 3 are strict in practice, their identifying power is not very strong in the sense that the parameters are just identified.

All the restrictions we have considered so far require that the period utility functions associated with one alternative (\( \mu_0^1, \ldots, \mu_{T-1}^0 \)) are known. In practice, researchers just normalize the period utility associated with one alternative to be a constant, which is usually zero, that is let \( \mu_0^T(S_t) = 0 \). But it has been gradually realized that normalizing period utility function is usually not innocuous
for predicting counterfactual policy effects (see e.g. Norets and Tang [2014] Arcidiacono and Miller [2014] Aguirregabiria and Suzuki [2014] Kalouptsidi, Scott, and Souza-Rodrigues [2015]). In appendix B, we show that normalization of period utility functions can usually bias the counterfactual policy predictions. To obtain unbiased counterfactual policy predictions, we have to identify the period utility functions associated with both alternatives up to an additive constant, that is invariant with respect to choices and states. More explicitly, let \((\mu_0^i, \mu_1^i)\) and \((\hat{\mu}_0^i, \hat{\mu}_1^i)\) both satisfy the identification system of equations (ID). Then we must have

\[
\mu_0^i - \mu_0^1 = \mu_1^i - \hat{\mu}_1^i = c_{\mu, t} \times 1_{d_s},
\]

for some real number \(c_{\mu, t} \in \mathbb{R}\). Here \(1_{d_s}\) is a \(d_s\)-dimensional vector of ones.

Below, we are going to provide two ways to circumvent the normalization. The first is to use the following excluded variables restriction.

**Restriction 4 (Excluded variables)**. The vector of observable states \(S_t\) has two parts \(X_t\) and \(Z_t\). Let \(S_t \equiv (X_t, Z_t)\), where \(X_t \in \mathcal{X} \equiv \{x_1, \ldots, x_{d_x}\}\) and \(Z_t \in \mathcal{Z} \equiv \{z_1, \ldots, z_{d_z}\}\). Assume \(\mu_1^i(X_t, Z_t) = \mu_1^i(X_t)\) and \(\mu_0^i(X_t, Z_t) = \mu_0^i(X_t)\) for any \((X_t, Z_t)\). For expositional simplicity, assume \(S = \mathcal{X} \times \mathcal{Z}\), so that \(d_s = d_x \times d_z\). In particular, let\(^\text{11}\)

\[
S \equiv \text{vec} \begin{bmatrix}
(x_1, z_1) \\
(x_2, z_1) \\
\vdots \\
(x_1, z_{d_z}) \\
(x_2, z_{d_z}) \\
\vdots \\
(x_1, z_1) \\
(x_2, z_1) \\
\vdots \\
(x_{d_x}, z_1) \\
(x_{d_x}, z_{d_z})
\end{bmatrix}.
\]

Notice that under the above restriction, \(\mu_s^i \equiv (\mu_1^i(x_1), \ldots, \mu_1^i(x_{d_x}))^T\) becomes \(d_s\)-dimensional vector. The above restriction is satisfied in our female labor force participation example, where \(S_t \equiv (husb_t, xp_t, edu, kid_t, xp_H^t, edu_H^t)\) with \(X_t = (husb_t, xp_t, edu, kid_t)\) and \(Z_t = (xp_H^t, edu_H^t)\). In general, given a set of state variables \(X_t\) that affect period utility, one may search for \(Z_t\) by finding the factors that could affect \(X_{t+1}\) but does not affect period utility given \(X_t\).

For example, in Rust’s [1987] bus engine replacement applications, \(X_t\) is the accumulated mileage of the bus. Then \(Z_t\) could be the characteristics of the bus’ route, which will affect the bus’ mileage in the next period, but not the current maintenance cost given the current mileage.

To see the identifying power of restriction 4, it is instructive to consider its application in the stationary dynamic programming problem first. When the agent’s dynamic programming problem is stationary and the excluded variable

\(^{11}\text{For } d_s = d_z = 2, \text{this means } S \equiv ((x_1, z_1), (x_1, z_2), (x_2, z_1), (x_2, z_2)).\)
restriction [4] holds, the linear system equation (ID) becomes

\[
\begin{align*}
\mu^{1/0}(X) + \delta E^{1/0} \{v(X', Z')|X, Z\} = \phi \{p(X, Z)\}, \\
v(X, Z) - \mu^0(X) - \delta E^0 \{v(X', Z')|X, Z\} = \psi \{p(X, Z)\}, \\
\forall S \in S.
\end{align*}
\]

We now have \(2d_x + ds + 1\) unknowns and \(2ds\) equations. So if \(ds \geq 2d_x + 1\), we may be able to identify the structural parameters. In particular, when \(ds = dx \times dz\), the order condition \(ds \geq 2d_x + 1\) would be satisfied as long as \(dz \geq 3\). To phrase the rank conditions precisely, we have to define some matrices first. These matrices will also be used in describing our estimators. Define a \(\{dx \times (dz - 1)\}\)-by-\(ds\) matrix \(M\),

\[
M \equiv I_{dx} \otimes \begin{bmatrix} 1 & -1 & \cdots & -1 \\ \vdots & \ddots & \ddots & \vdots \\ 1 & -1 \end{bmatrix}_{(dz-1) \times dz},
\]

a \([2 \times \{dx \times (dz - 1)\}]\)-by-\(ds\) matrix \(A\),

\[
A \equiv \begin{bmatrix} \delta MF^{1/0} \\ M(I - \delta F^0) \end{bmatrix},
\]

a \([2 \times \{dx \times (dz - 1)\}]\)-dimensional vector \(b\),

\[
b \equiv \begin{bmatrix} M\phi(p) \\ M\psi(p) \end{bmatrix},
\]

and a \(dx\)-by-\(ds\) matrix \(W\),

\[
W \equiv I_{dx} \otimes (dz^{-1} \times 1_{dz})^T.
\]

Here \(p = (p(x_1, z_1), \ldots, p(x_1, zd), \ldots, p(x_{dx}, z_1), \ldots, p(x_{dx}, zd))^T\), and the two vectors \(\phi(p)\) and \(\psi(p)\) are defined by letting the ith entry of \(\phi(p)\) and \(\psi(p)\) be \(\phi(p_i)\) and \(\psi(p_i)\), respectively, where \(p_i\) is the ith element of \(p\).

**Proposition 5** (Identification with restriction [3(i)] and [3(ii)] and [4]. In addition to assumption [2] and [3], suppose restriction [3(i)] and [3(ii)] and [4] also holds. Define \(v_+ = A^+b\), where \(A^+\) is the Moore-Penrose pseudoinverse of matrix \(A\) in equation (3.11). If rank \(A = ds - 1\), we have the following conclusions.
(i) The difference between period utility functions \( \mu^{1/0} \) is identified, and
\[
\mu^{1/0} = W \left\{ \phi(p) - \delta F^{1/0} \psi_+ \right\}.
\]

(ii) The identified set of \( (\mu^0, \mu^1) \) is
\[
\left\{ \mu^0_+ + c_\mu \times 1_{d_s}, \mu^1_+ + c_\mu \times 1_{d_s} : c_\mu \in \mathbb{R} \right\},
\]
where
\[
\mu^0_+ = W \left\{ (I - \delta F^0) \psi_+ - \psi(p) \right\},
\mu^1_+ = \mu^0_+ + \mu^{1/0}.
\]

Proof. The proof is simply a procedure of solving equation (3.9), which can be expressed as the following matrix form,
\[
\begin{align*}
\{ \mu^{1/0} \otimes 1_{d_s} + \delta F^{1/0} v = \phi(p) \\
v - \mu^0 \otimes 1_{d_s} - \delta F^0 v = \psi(p)
\end{align*}
\]
where “\( \otimes \)” denote the Kronecker product.

It follows from \( M(\mu^{1/0} \otimes 1_{d_s}) = M(\mu^0 \otimes 1_{d_s}) = 0_{d_s} \) that
\[
\begin{bmatrix}
\delta M F^{1/0} \\
M(I - \delta F^0)
\end{bmatrix} v = \begin{bmatrix}
M \phi(p) \\
M \psi(p)
\end{bmatrix},
\]
which is \( A v = b \) using the defined matrix \( A \) and vector \( b \) in equation (3.11) and (3.12), respectively. The rank of the coefficient matrix \( A \) cannot exceed \( d_s - 1 \) because the sum of its columns is a zero vector. So the solution for \( v \) will not be unique. But if rank \( A = d_s - 1 \), the solution set of \( v \) from equation (3.14) is \( \{ v_+ + c \times 1_{d_s} : \text{for all } c \in \mathbb{R} \} \), by lemma A.1. For each \( v = v_+ + c \times 1_{d_s} \), we then have a corresponding \( \mu^0 \otimes 1_{d_s} = (I - \delta F^0) v + c_\mu - \psi(p) \) or \( \mu^0 = W \{ (I - \delta F^0) v + c_\mu - \psi(p) \} \) with \( c_\mu = (1 - \delta) c \), and \( \mu^{1/0} = W \{ \phi(p) - \delta F^{1/0} v_+ \} = W \{ \phi(p) - \delta F^{1/0} \psi_+ \} \) for \( F^{1/0} 1_{d_s} = 0_{d_s} \). Letting \( \mu^0_+ = W \{ (I - \delta F^0) v_+ - \psi(p) \} \), the solution set, or equivalently the identified set, of \( \mu^0 \) is \( \{ \mu^0_+ + c_\mu \times 1_{d_s} : c_\mu \in \mathbb{R} \} \). Since \( \mu^{1/0} \) is identified, the identified set of \( (\mu^0, \mu^1) \) is as stated in the proposition.

It is remarkable that Proposition B.1 and B.2 in section 3 show that the counterfactual policy effects will be pointly identified as long as the period utility

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function is identified up to an additive constant, that is identical for all states and all alternatives. So the partial identification conclusion in Proposition 5 is indeed enough for making counterfactual policy predictions. The rank condition that \( \text{rank } A = d_s - 1 \) is important. We add additional comments for the rank condition in remark 2.

The same arguments in the proof of Proposition 5 can be extended to identify the general nonstationary DPDC models. Applying the exclusion restriction 4, equation (ID) becomes

\[
(3.15) \quad \begin{cases} 
\mu_t^{1/0}(X_t) + \delta_t \mathbb{E}^{1/0}_{t+1} \{v_{t+1}(X_{t+1}, Z_{t+1})|X_t, Z_t\} = \phi(p_t(X_t, Z_t)) \\
\nu_t(X_t, Z_t) - \mu_t^0(X_t) - \delta_t \mathbb{E}^0_{t+1} \{v_{t+1}(X_{t+1}, Z_{t+1})|X_t, Z_t\} = \psi(p_t(X_t, Z_t))
\end{cases}
\]

for all \((X_t, Z_t) \in \mathcal{X} \times \mathcal{Z}\) and \(t = 1, \ldots, T - 1\). There are \(2 \times (T - 1) \times d_x + T \times d_s + (T - 1)\) unknowns and \(2 \times (T - 1) \times d_s\) equations. So as long as \(d_z \geq 3\) and \(T \geq 3\), we would have more equations than unknowns. It is remarkable that when \(T < 3\), the order condition always fail regardless of the value of \(d_z\). When \(T = 2\), we have \(2d_s\) equations but \(2d_x + 2d_s + 1\) unknowns. So in order to identify a nonstationary DPDC model without imposing normalization restriction, one must have at least 3 consecutive periods data (the requirement for being “consecutive” will be clarified after the proof of Proposition 6). Again, we need some matrices to precisely state the rank conditions and the identified set of parameters. Let \(A_t\) be a \(\{3 \times d_x \times (d_z - 1)\}\)-by-(2 \(\times d_s\) matrix and \(b\) be a \(\{3 \times d_x \times (d_z - 1)\}\)-dimensional vector:

\[
(3.16) \quad A_t \equiv \begin{bmatrix} 
MF_t^{1/0} & M F_t^{1/0} \\
M & -MF_{t+1}^0
\end{bmatrix},
\]

\[
(3.17) \quad b_t \equiv \begin{bmatrix} 
\delta_{t-1}^{-1} M \phi(p_{t-1}) \\
M \phi(p_t) \\
M \psi(p_t)
\end{bmatrix}.
\]

**Proposition 6** (Identification with restriction 4 and known discount factors). In addition to assumption 2 and 3, suppose restriction 4 holds, the discount factors are known and \(T \geq 3\). For \(t = 2, \ldots, T - 1\), let \(\bar{v}_{t+,} = A_{t, h} b_t\) and \(\bar{v}_{t+,+} = A_{t, l}^+ b_t\), where \(A_{t, h}\) and \(A_{t, l}^+\) are the \(d_s\)-by-(3 \(\times d_x \times (d_z - 1))\) matrices formed by the first and last \(d_s\) rows of matrix \(A_t^+\) (the Moore-Penrose pseudoinverse).
verse of $A_t$), that is

\[(3.18) \quad A_t^+ = \left[ \begin{array}{c} A_{t,h}^+ \\ A_{t,t}^+ \end{array} \right].\]

If rank $A_t = 2d_s - 2$, we have the following conclusions.

(i) The difference between period utility functions $\mu_{t-1}^{1/0}$, $\mu_t^{1/0}$ are identified.

And we have

\[
\mu_{t-1}^{1/0} = W \left\{ \phi(p_{t-1}) - \delta_{t-1} F_{t}^{1/0} \bar{\psi}_{t,+} \right\},
\]

\[
\mu_t^{1/0} = W \left\{ \phi(p_t) - F_{t+1}^{1/0} \bar{\psi}_{t+1,+} \right\}.
\]

(ii) The identified set of $(\mu_t^0, \mu_t^1)$ is

\[
\left\{ \bar{\psi}_{t,+} + c_{\mu,t} \times 1_{d_s}, \bar{\psi}_{t,+} + c_{\mu,t} \times 1_{d_s} : c_{\mu,t}, c_{\mu,t+1} \in \mathbb{R} \right\},
\]

where

\[
\mu_{t,+}^0 = W \left\{ \bar{\psi}_{t,+} - F_{t+1}^0 \bar{\psi}_{t+1,+} - \psi(p_t) \right\},
\]

\[
\mu_{t,+}^1 = \mu_{t,+}^0 + \mu_{t}^{1/0}.
\]

Proof. For each period $t = 2, \ldots, T - 1$, we have the following equations as a part of equation \[(3.15)\],

\[
\left\{ \begin{array}{l}
\mu_t^{1/0}(X_t) + \mathbb{E}_t^{1/0} \{ \bar{v}_{t+1}(X_{t+1}, Z_{t+1}) | X_t, Z_t \} = \phi(p_t(X_t, Z_t)) \\
\mu_{t-1}^{1/0}(X_{t-1}) + \delta_{t-1} \mathbb{E}_{t-1}^{1/0} \{ \bar{v}_{t}(X_{t-1}, Z_{t-1}) | X_{t-1}, Z_{t-1} \} = \phi(p_{t-1}(X_{t-1}, Z_{t-1})) \\
\bar{v}_t(X_t, Z_t) - \mu_t^0(X_t) - \mathbb{E}_{t+1}^0 \{ \bar{v}_{t+1}(X_{t+1}, Z_{t+1}) | X_{t+1}, Z_{t+1} \} = \psi(p_t(X_t, Z_t))
\end{array} \right.
\]

for all $(X_{t-1}, Z_{t-1}), (X_t, Z_t) \in \mathcal{X} \times \mathcal{Z}$. Here $\bar{v}_{t+1}(X_{t+1}, Z_{t+1}) \equiv \delta_1 \bar{v}_{t+1}(X_{t+1}, Z_{t+1})$. To identify $\mu_t^0$, it suffices to identify $\bar{v}_t(S_t)$ and $\bar{v}_{t+1}(S_{t+1})$. Rewrite the above three equations as the following matrix form:

\[(3.19) \quad \left\{ \begin{array}{l}
\bar{\mu}_t^{1/0} \otimes 1_{d_s} + F_{t+1}^{1/0} \bar{\psi}_{t+1} = \phi(p_t) \\
\bar{\mu}_{t-1}^{1/0} \otimes 1_{d_s} + \delta_{t-1} F_t^{1/0} \bar{\psi}_t = \phi(p_{t-1}) \\
\bar{\psi}_t - \bar{\mu}_t^0 \otimes 1_{d_s} - F_{t+1}^0 \bar{\psi}_{t+1} = \psi(p_t)
\end{array} \right.\]

Using the condition that $M(\mu_t^{1/0} \otimes 1_{d_s}) = M(\mu_{t-1}^{1/0} \otimes 1_{d_s}) = M(\mu_t^0 \otimes 1_{d_s}) = 0_{d_s}$, we have

\[(3.20) \quad A_t \left[ \begin{array}{c}
\bar{\psi}_t \\
\bar{\psi}_{t+1}
\end{array} \right] = b_t,\]

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where the coefficient matrix $A_t$ and vector $b_t$ are defined in equation (3.16) and (3.17), respectively. If $\text{rank } A_t = 2 \times (d_s - 1)$, we know from lemma A.2 that the solution set of equation (3.20) is that
\[
\left\{ \begin{array}{c}
\bar{v}_{t,+} + c_t \times 1_{d_s} \\
\bar{v}_{t+1,+} + c_{t+1} \times 1_{d_s}
\end{array} \right. : c_t, c_{t+1} \in \mathbb{R}.
\]

We then have the conclusion stated in the proposition by using the same arguments in the proof of Proposition 5.

To see why we need to consecutive periods, suppose we have data on the decision process in periods $t - 2$, $t$ and $t + 1$, which are not consecutive. Then we have equations (3.21)
\[
\begin{align*}
m_t^{1/0} \otimes 1_{d_s} &+ F_{t+1/0}^{1/0} \bar{v}_{t+1} = \phi(p_t) \\
m_{t-2}^{1/0} \otimes 1_{d_s} + \delta_{t-2} F_{t-1/0}^{1/0} \bar{v}_{t-1} = \phi(p_{t-2}) \\
v_{t} - m_t^{0} \otimes 1_{d_s} - F_{t+1/0}^{0} \bar{v}_{t+1} = \psi(p_t)
\end{align*}
\]

This system of equations is not well defined because $F_{t-1/0}^{1/0}$ is not identified, as we don’t have data about in period $t - 1$. Even when $F_{t-1}^{1/0}$ is known, we still cannot identify the parameters, such as $m_{t-2}^{1/0}$, $m_t^{1/0}$ and $m_t^{0}$, because unlike equation (3.19), there are three unknown ex ante value functions $\bar{v}_{t+1}$, $\bar{v}_{t}$ and $\bar{v}_{t-1}$, giving rise to $3d_s$ unknowns, in equation (3.21). So the order condition won’t hold. However, the requirement for three consecutive periods data can indeed be relaxed, if we know the transition matrices for each period or the transition matrices are time invariant. Note that for equation (3.19), we need period $t + 1$ only because we need to know $F_{t+1}^{1/0}$. So if $F_{t+1}^{1/0}$ is known, equation (3.19) is well defined, and the proposition will follow.

We want to emphasize again that although the period utility functions are identified up to an additive constant. The counterfactual policy effects are pointly identified for Proposition B.1 and B.2. When the panel data have length greater than 4, we can also identify the discount factors.

**Proposition 7** (Identification of discount factors with restriction 4 and $T \geq 4$). Suppose the conditions of Proposition 6 hold with $T \geq 4$. If $\text{rank } F_t^{1/0} = d_s - 1$, and the matrix $B_t$ define in equation (A.4) is of full rank, the discount factors $\delta_{t-2}$ and $\delta_{t-1}$ are identified.

In practice, the excluded state variable $Z_t$ could be time invariant. For example, in Rust’s (1987) bus engine replacement application, the excluded state
variable can be a permanent route characteristics of the bus. Recently, Fang and Wang (2015) use the excluded variables to identify hyperbolic discounting parameters. In their application of mammography decisions, the excluded variables include categorical variables like education and race, which do not change over time. When the excluded variable $Z_t$ is time invariant, the conclusion of Proposition 6 and 7 hold under a different rank condition.

**Proposition 8.** [Identification with permanent excluded state variables] Suppose the excluded state variable $Z_t$ is time invariant, that is $Z_t = Z_{t'}$ for any $1 \leq t < t' \leq T$. If the conditions of Proposition 6 and 7 hold with rank condition $\text{rank} A_t = 2d_s - d_z - 1$, the conclusions of the two propositions hold.

When the excluded variables $Z_t$ do not exist, an alternative way to circumvent the normalization of period utility function is to assume that the period utility functions are time invariant but the state transition matrices are time varying. Define a $3d_s$-by-$3d_s$ matrix $C_t$ and a $3d_s$-dimensional vector $h_t$:

$$C_t \equiv \begin{bmatrix} F_{t+1}^{1/0} & -\delta_{t-1}F_{t}^{1/0} & 0 \\ \delta_{t-1}F_{t}^{1/0} & -\delta_{t-2}F_{t-1}^{1/0} & 0 \\ -F_{t+1}^{0} & I + \delta_{t-1}F_{t}^{0} & -I \end{bmatrix} \quad \text{and} \quad h_t = \begin{bmatrix} \phi(p_t) - \phi(p_{t-1}) \\ \phi(p_{t-1}) - \phi(p_{t-2}) \\ \psi(p_t) - \psi(p_{t-1}) \end{bmatrix}.$$  

**Proposition 9** (Identification with stable period utilities and unstable transition matrices). Suppose assumptions 2 and 3 hold, and $T \geq 4$. For each $t \geq 3$, suppose the discount factors $\delta_t$, $\delta_{t-1}$ and $\delta_{t-2}$ are known, $\mu_t^d(S) = \mu_{t-1}^d(S) = \mu_{t-2}^d(S) = \mu^d(S)$, and rank $C_t = 3d_s - 2$.

(i) The difference between period utility functions $\mu^{1/0}$ is identified, and

$$\mu^{1/0} = 3^{-1} \times \left\{ \phi(p_t) + \phi(p_{t-1}) + \phi(p_{t-2}) - \begin{bmatrix} F_{t+1}^{1/0} & \delta_{t-1}F_{t}^{1/0} & \delta_{t-2}F_{t-1}^{1/0} \end{bmatrix} C_t^+ h_t \right\}.$$  

(ii) The identified set of $(\mu^0, \mu^1)$ is

$$\left\{ \mu^0_+ + c_\mu \times 1_{d_s}, \mu^1_+ + c_\mu \times 1_{d_s} : c_\mu \in \mathbb{R} \right\},$$

where

$$\mu^0_+ = 2^{-1} \left\{ \begin{bmatrix} -F_{t+1}^0 & (I_{d_s} - \delta_{t-1}F_{t}^0) & I_{d_s} \end{bmatrix} C_t^+ h_t - \psi(p_t) - \psi(p_{t-1}) \right\},$$

$$\mu^1_+ = \mu^0_+ + \mu^{1/0}.$$
Remark 1 (Extension to multinomial choices). The above arguments can be extended by using the general Hotz-Miller inversion formulas \cite{Hotz and Miller1993}. Suppose the choice set \{0, 1, \ldots, J\} contains \(J + 1\) alternatives. By the Hotz-Miller’s formula, there exists \(\phi^j : j = 1, \ldots, J\) and \(\psi\) such that

\[
\begin{cases}
  v^j_t(S_t) - v^0_t(S_t) = \phi^j\{p_t(S_t)\} \\
  v_t(S_t) - v^0_t(S_t) = \psi\{p_t(S_t)\},
\end{cases}
\]

where \(p_t(S_t) \equiv (\mathbb{P}(Y_t = 1|S_t), \ldots, \mathbb{P}(Y_t = J|S_t))^T\). Equation (1D) becomes

\[
\begin{cases}
  \mu^j_t(S_t) + \delta_t \mathbb{E}_t^{j/0}\{v_{t+1}(S_{t+1})|S_t\} = \phi^j\{p_t(S_t)\} \\
  v_t(S_t) - \mu^0_t(S_t) - \delta_t \mathbb{E}_t^{0/1}\{v_{t+1}(S_{t+1})|S_t\} = \psi\{p_t(S_t)\},
\end{cases}
\]

for all \(S_t \in \mathcal{S}, j = 1, \ldots, J\), and \(t = 1, \ldots, T - 1\), subject to \(\theta \in \Theta\).

More alternatives provide more information about the structural parameters. So the exact identification results for multinomial choices are slightly different from the above propositions, but the general idea is similar.

Remark 2 (Rank conditions). The rank conditions in Proposition 5 are clearly important. As a rule-of-thumb, these rank conditions require that the rows of the matrix \(MF_{t+1}^{1/0}\) do not have many zero vectors. To be concrete, take the rank condition in Proposition 5 for example. We have

\[
\text{rank}
\begin{bmatrix}
  \delta MF^{1/0} \\
  M(I - \delta F^0)
\end{bmatrix}
= \text{rank}\{M(I - \delta F^0)\}
+ \text{rank}\left\{(I_{dx} - P_{\{M(I - \delta F^0)\}^T})(\delta MF^{1/0})\right\}^T
\]

\[
= d_x - d_x + \text{rank}\left\{(I_{dx} - P_{\{M(I - \delta F^0)\}^T})(\delta MF^{1/0})\right\}^T
\]

\[
\equiv d_x - d_x + r.
\]

Here \(P_{\{M(I - \delta F^0)\}^T}\) is the projection matrix generated by matrix \(\{M(I - \delta F^0)\}^T\). To satisfy the rank condition, we wish have large \(r\). If there are many zero rows in \(MF^{1/0}\), \(r\) will be smaller than \(d_x - 1\). Each row of \(MF^{1/0}\) takes the form

\[
(f^{1/0}(s_1|x, z_j) - f^{1/0}(s_1|x, z_{j+1}), \ldots, f^{1/0}(s_{dx}|x, z_j) - f^{1/0}(s_{dx}|x, z_{j+1})),
\]

where \(f^{1/0}(s_k|x, z_j) \equiv f(s_k|x, z_j, 1) - f(s_k|x, z_j, 0)\). So the row will be zero if the choice does not change transition matrix (hence \(f^{1/0}(s|x, z) = 0\)) or the excluded variable \(Z\) does not affect the difference between the transition probabilities given the state variable \(X\) that affects period utility functions (hence
Magnac and Thesmar define a function $U(s|x_i, z_j) = f^{1/0}(s|x_i, z_{j+1}) = 0$.

**Remark 3** (Difference between our Proposition 3 and proposition 4 of Magnac and Thesmar [2002]). In a two period model, Magnac and Thesmar (2002, Proposition 4) also consider the identification of discount factor using exclusion condition. But their exclusion is very different from ours. In two period case $(T = 2)$, their exclusion is that there exists $s_1, s_1' \in S$, such that

$$
\text{(3.23)} \quad \mu_1^{1/0}(s_1) + \delta_1 E_2^{1/0}\{v_2^0(S_2)|s_1\} = \mu_1^{1/0}(s_1') + \delta_1 E_2^{1/0}\{v_2^0(S_2)|s_1'\}.
$$

Magnac and Thesmar define a function $U(s) = \mu_1^{1/0}(s) + \delta_1 E_2^{1/0}\{v_2^0(S_2)|s\}$, so their exclusion condition (3.23) can be written as $U(s_1) = U(s_1')$. Our exclusion in two period case says that

$$
\mu_1^{1/0}(s_1) = \mu_1^{1/0}(s_1').
$$

Our exclusion is solely based on period utility functions, but Magnac and Thesmar’s involves not only period utility functions but also ASVF $v_2^0(S_2)$. It is easy to see how to identify discount factor $\delta_1$ from equation (3.23). Plugging $v_2^0(S_2) = v_2(S_2) - \psi\{p_2(S_2)\}$ into equation (3.23), the left-hand-side becomes

$$
U(s_1) = \mu_1^{1/0}(s_1) + \delta_1 E_2^{1/0}\{v_2(S_2)|s_1\} - \delta_1 E_2^{1/0}[\psi\{p_2(S_2)\]|s_1] = \phi\{p_1(s_1)\} - \delta_1 E_2^{1/0}[\psi\{p_2(S_2)\]|s_1].
$$

Then the condition (3.23) says

$$
\phi\{p_1(s_1)\} - \delta_1 E_2^{1/0}[\psi\{p_2(S_2)\]|s_1] = \phi\{p_1(s_1')\} - \delta_1 E_2^{1/0}[\psi\{p_2(S_2)\]|s_1'],
$$

hence $\delta_1$ is identified if $E_2^{1/0}[\psi\{p_2(S_2)\]|s_1] \neq E_2^{1/0}[\psi\{p_2(S_2)\]|s_1'].$

### 4 Linear Estimation

All identification results in the previous section are constructively proved by solving the identified structural parameters from the linear system of equations (ID). And the solutions have closed form. It is natural to estimate these identified structural parameters by using the “estimated” closed form solutions. The estimation proceeds in two steps. The first step is to estimate the unknown CCP $\{p_t(S_t) : t = 1, \ldots, T - 1\}$ and transition matrix $\{F^d_t : t = 1, \ldots, T - 1\}$.

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12 The notation “$U$” makes people think $U(s)$ is the period utility, hence confuse their exclusion with ours.
0, 1). Let \( \hat{\mu}_t(S_t) \) and \( \hat{F}^d_{t+1} \) be the estimates of the CCP \( p_t(S_t) \) and transition matrix \( F^d_{t+1} \) for each alternative \( d \) and each period \( t \). The second step is to estimate the structural parameters using the closed form solutions from different identification restrictions.

We focus on the case using the restriction of excluded variables (Proposition 8) with known discount factors. It follows from Proposition 8 that for each \( t = 2, \ldots, T - 1 \), we have

\[
\begin{align*}
\hat{\mu}^{1/0}_{t-1} &= W \left\{ \phi(\hat{\mu}_{t-1}) - \delta_{t-1}\hat{F}^1_{t-1} \hat{\mu}_{t-1} \right\}, \\
\hat{\mu}^{1/0}_t &= W \left\{ \phi(\hat{\mu}_t) - \hat{F}^1_t \hat{\mu}_{t+1} \right\}, \\
\hat{\mu}^0_{t+1} &= W \left\{ \hat{\mu}_{t+1} - F^0_{t+1} \hat{\mu}_{t+1} - \psi(\hat{\mu}_t) \right\},
\end{align*}
\]

where \( W \equiv I_{d_t} \otimes (d_t^{-1/2} \times 1_{d_t})^T \). It is tricky to describe \( \hat{\mu}_{t+1} \) and \( \hat{\mu}_{t+1} \). For \( t = T - 1 \), we can have only \( \hat{\mu}_{t+1} = A^t_{t+1} b_t \) and \( \hat{\mu}_{t+1} = A^t_{t+1} b_t \) for \( t = 2, \ldots, T - 2 \) (if \( T \geq 4 \)), however, we can have \( \hat{\mu}_{t+1} = A^t_{t+1} b_t \) or \( \hat{\mu}_{t+1} = \delta_{t-1} A^t_{t+1} b_t \), and \( \hat{\mu}_{t+1} = A^t_{t+1} b_t \) or \( \hat{\mu}_{t+1} = \delta_{t-1} A^t_{t+1} b_t \). It turns out that letting \( \hat{\mu}_{t+1} = A^t_{t+1} b_t \) and \( \hat{\mu}_{t+1} = \delta_{t-1} A^t_{t+1} b_t \), for \( t = 2, \ldots, T - 2 \), can substantially improve the estimation efficiency (in terms of smaller mean squared error) of \( \hat{\mu}^{1/0}_t \) and \( \hat{\mu}^0_{t+1} \).

The efficiency improvement results from the structure of matrix \( A_t \). Detailed explanation can be found in the supplemental material.

For \( t = 2, \ldots, T - 1 \), we have estimators

\[
\begin{align*}
\hat{\mu}^{1/0}_{t-1} &= W \left\{ \phi(\hat{\mu}_{t-1}) - \delta_{t-1}\hat{F}^1_{t-1} \hat{\mu}_{t-1} \right\}, \\
\hat{\mu}^{1/0}_t &= W \left\{ \phi(\hat{\mu}_t) - \hat{F}^1_t \hat{\mu}_{t+1} \right\}, \\
\hat{\mu}^0_{t+1} &= W \left\{ \hat{\mu}_{t+1} - F^0_{t+1} \hat{\mu}_{t+1} - \psi(\hat{\mu}_t) \right\},
\end{align*}
\]

where \( \hat{\mu}_{t+1} = A^t_{t+1} b_t \) for each \( t = 2, \ldots, T - 1 \), and

\[
\hat{\mu}_{t+1} = \begin{cases} 
\delta_t A^t_{t+1} b_t, & t = 2, \ldots, T - 2, \\
A^t_{t+1} b_t, & t = T - 1.
\end{cases}
\]

Here \( \hat{\mu}_{t-1}, \hat{\mu}_t, A^t_{t+1}, \hat{\mu}_{t+1}, b_t, \hat{F}^1_{t+1}, F^0_{t+1} \), and \( F^1_{t+1} \) are the estimators of their counterparts. Note that these estimators have closed-form, and their computation is easy. So their variance can be easily be obtained from bootstrap.

Remark 4 (Data requirement). To implement the estimators in equation (4.1), we need \( \hat{\mu}_{t-1}, \hat{\mu}_t, \hat{F}^1_t, F^0_{t+1}, F^1_{t+1} \). So if the transition matrices are
unknown, we need panel data that cover the decision process over periods \( t - 1, t \) and \( t + 1 \). But if \( F^d_t = F^d_{t+1} \) for both \( d = 0, 1 \), or simply if \( F^0_{t+1} \) and \( F^1_{t+1} \) are known, we only need panel data to cover periods \( t - 1 \) and \( t \). If the agent’s dynamic programming problem is stationary, and the transition matrix is known, we can estimate the period utility \( \tilde{\mu}^0 \) and \( \tilde{\mu}^1 \) even from cross-sectional data using Proposition 5.

**Remark 5 (Choice of the number of excluded states)**. Note that the matrix 
\[
W \equiv I_{d_z} \otimes \left( d_z^{-1} \times 1_{d_z} \right)^T
\]
depends on the number of excluded states \( d_z \). When \( d_z \) becomes larger, \( W \) becomes “smaller” (more precisely, the spectral norm of \( W \) is \( d_z^{-1} \)), and the variance of the above estimators is expected to be smaller if everything else is unchanged. But a larger \( d_z \), hence a larger state space, makes the estimation of the CCP and transition matrices less precise. To see this issue, take the estimator 
\[
\hat{\mu}_{t-1}^{1/0}
\]
for example, and suppose the variance matrix of 
\[
\phi \left( \hat{\mu}_{t-1}^{1/0} - \delta_{t-1} F_t^{1/0} \tilde{A}_t^+ b_t \right)
\]
is a \( d_s \)-by-\( d_s \) diagonal matrix \( D \). Then the trace of the variance matrix of \( \hat{\mu}_{t-1}^{1/0} \) is 
\[
\text{tr}(D)/d_z^2.
\]
In practice, one may use the formula 
\[
\text{tr}\{\text{Var}(\hat{\mu}_{t-1}^{1/0})\} = \text{tr}(D)/d_z^2
\]
as a guidance for selecting \( d_z \).

**Remark 6 (Estimation with large state space)**. When the state space is large, it is likely that matrix \( A_t \) will be ill-conditioned creating large variance for the estimates in equation (4.1). One way to deal with this large space issue is to implement the estimators in equation (4.1) first, then refine the estimates using minimum distance estimators by assuming that \( \mu_t^{1/0}(X_t) \) and \( \mu_t^0(X_t) \) have parametric specification or the vectors \( \hat{\mu}_t^{1/0} \) and \( \hat{\mu}_t^0 \) are obtained by interpolating smooth functions \( \mu_t^{1/0}(X_t) \) and \( \mu_t^0(X_t) \) at the \( d_x \) discrete points in \( \mathcal{X} \). Taking \( \mu_t^{1/0}(X_t) \) for example, we can approximate

\[
\mu_t^{1/0}(X_t) \approx \sum_{\ell=1}^{K} \alpha_{t,\ell} \rho_{t}(X_t),
\]

where \( \{\rho_{t}(\cdot) : \ell = 1, 2, \ldots\} \) is a sequence of orthonormal polynomial basis functions. Given the estimates \( \hat{\mu}_t^{1/0} \) from equation (4.1), we can estimate \( \alpha_t^{1/0} = (\alpha_{t,1}^{1/0}, \ldots, \alpha_{t,K}^{1/0}) \) by least square. Denote \( \hat{\alpha}_t^{1/0} \) the obtained estimates. Then a refined estimate of \( \mu_t^{1/0} \) will be

\[
\hat{\mu}_t^{1/0} = \sum_{\ell=1}^{K} \hat{\alpha}_{t,\ell} \rho_{t}(X_t).
\]

When the space is so large that the direct computation of the Moore-Penrose
pseudoinverse $A_t^+$ in equation 4.1 becomes difficult, we may consider smoothing the ex ante value function $v_t(X_t, Z_t)$ and $\bar{v}_{t+1}(X_{t+1}, Z_{t+1})$ in the first place. Let $\{r_t(\cdot) : \ell = 1, 2, \ldots \}$ be a sequence of orthonormal polynomial basis functions. Similarly, let

$$v_t(x_i, z_j) \approx \sum_{\ell=1}^{K_\ell} \beta_{t,\ell} r_\ell(x_i, z_j),$$

$$\bar{v}_{t+1}(x_i, z_j) \approx \sum_{\ell=1}^{K_\ell} \beta_{t+1,\ell} r_\ell(x_i, z_j),$$

for some $K \leq d$. Define a $d \times K_v$ matrix $R$ as follows,

$$R = \begin{bmatrix} r_1(x_1, z_1) & \cdots & r_{K_v}(x_1, z_1) \\ \vdots & \ddots & \vdots \\ r_1(x_{d_x}, z_{d_z}) & \cdots & r_{K_v}(x_{d_x}, z_{d_z}) \end{bmatrix},$$

and let $\tilde{R} \equiv 1_{2 \times 1} \otimes R$. Define two $K_v$-dimensional vectors $\beta_t = (\beta_{t,1}, \ldots, \beta_{t,K_v})^T$ and $\beta_{t+1} = (\beta_{t+1,1}, \ldots, \beta_{t+1,K_v})^T$. We then have

$$A_t \tilde{R} \begin{bmatrix} \beta_t \\ \beta_{t+1} \end{bmatrix} \approx b_t.$$

Then an estimate of $\hat{v}_t$ and $\hat{\bar{v}}_{t+1}$ can be formed by letting

$$\begin{bmatrix} \hat{v}_t \\ \hat{\bar{v}}_{t+1} \end{bmatrix} = \tilde{R}(A_t \tilde{R})^+ b_t.$$

With $\hat{v}_t$ and $\hat{\bar{v}}_{t+1}$, the period utility functions can be estimated. A practically important issue is about the choice of $K_\mu$ and $K_v$. A formal treatment of this issue requires some extension of the adaptive linear estimators (see Chapter 5 and 6 of [Johnstone 2013]) and is beyond the scope of this paper.

**Remark 7 (MLE is still applicable).** Using equation 3.15, one can still apply MLE to estimate the model. We have

$$p_{T-1}(x_{T-1}, z_{T-1}) = \phi^{-1} \left[ \mu_{T-1}^{1/0}(x_{T-1}) + \delta_{T-1} E_{T}^{1/0} \{ v_T(X_T, Z_T) | x_{T-1}, z_{T-1} \} \right],$$

and

$$p_t(x_t, z_t) = \phi^{-1} \left[ \mu_t^{1/0}(x_t) + \delta_t E_{t+1}^{1/0} \{ v_{t+1}(x_{t+1}, z_{t+1}) | x_t, z_t \} \right],$$
for \( t < T - 1 \), where 
\[
\tilde{v}_t = \mu_t^0 + \delta_t F_{t+1}^0 \tilde{v}_{t+1} - \psi(p_t)
\]
is recursively defined. Depending on the size of the state space, one may or may not parameterize the period utility functions and the ex ante value function \( v_T(S_T) \). This does not affect the above representation of the CCP. Two things are remarkable in using MLE. First, the estimation of state transition densities and the estimation of the period utility functions and discount factors can be separated. Suppose data are \( \{x_{it}, z_{it}, y_{it} : i = 1, \ldots, n, t = 1, \ldots, T\} \). After estimating the state transition matrices, one can estimate the period utility functions and discount factors by maximizing 
\[
\sum_{i=1}^{n} \sum_{t=1}^{T} \left[ y_{it} \ln p_t(x_{it}, z_{it}) + (1 - y_{it}) \ln \left( 1 - p_t(x_{it}, z_{it}) \right) \right]
\]
alone, because Proposition 6 has guaranteed the identification of period utility and discount factors from the CCP only. Second, the length of panel data should be greater than 3 for known discount factors and 4 for unknown discount factors. Otherwise, the identification of the model is not guaranteed.

5 Numerical Studies

In the numerical experiments, we consider a single \( X_t \) and a single excluded variable \( Z_t \), and let \( S_t = (X_t, Z_t) \). The decision horizon is \( T_\ast = 10 \). The structural period utility functions are
\[
\begin{cases}
  \mu_t^1(X_t) = X_t,
  \\
  \mu_t^0(X_t) = \frac{1}{2} + \frac{X_t}{3},
\end{cases}
\]
the discount factors \( \delta_t \) are 0.8 for all periods, and the utility shocks \( (\varepsilon_0^t, \varepsilon_1^t) \) are generated from type 1 extreme value distribution.

The support \( \mathcal{X} \equiv \{x_1, \ldots, x_{d_x}\} \) of \( X_t \) are the \( d_x \) cutting points that split the interval \( [0, 2] \) into \( d_x - 1 \) equally spaced subintervals. And the support \( \mathcal{Z} \equiv \{z_1, \ldots, z_{d_z}\} \) of \( Z_t \) are the cuttings points that split \( [0, 2] \) into \( d_z - 1 \) equally spaced subintervals. Let the state space \( S \equiv \mathcal{X} \times \mathcal{Z} \), and \( d_s = d_x \times d_z \). The observable states \( S_t = (X_t, Z_t) \) follows a homogenous controlled first-order Markov chain. Let \( F^d \) be the time invariant \( d_x \times d_s \) transition matrix describing the transition probability law from \( S_t \) to \( S_{t+1} \) given the discrete choice \( Y_t = d \). The transition matrix \( F^d \) is randomly generated subjecting to the sparsity restriction that there are at most \( m_s \) number of states that can be reached in the next period. The data for estimation are \( \{(x_{it}, z_{it}, y_{it}) : i = 1, \ldots, n, t = 1, \ldots, 4\} \). So we observe only the first four periods of the dynamic

13 Ignoring the state transition densities in the log likelihood function can lose some efficiency in estimating period utility functions.
Table 1: Estimation of Period Utility Functions: $d_x = 3, d_z = 4$

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$T = 3$</th>
<th>$T = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_1^{1/0}(x_1)$</td>
<td>$&lt; 1e-2$</td>
<td>$&lt; 1e-2$</td>
</tr>
<tr>
<td>$\mu_1^{1/0}(x_2)$</td>
<td>$&lt; 1e-3$</td>
<td>$&lt; 1e-2$</td>
</tr>
<tr>
<td>$\mu_1^{1/0}(x_3)$</td>
<td>$&lt; 1e-2$</td>
<td>$&lt; 1e-2$</td>
</tr>
<tr>
<td>$\mu_2^{1/0}(x_1)$</td>
<td>$&lt; 1e-2$</td>
<td>$&lt; 1e-3$</td>
</tr>
<tr>
<td>$\mu_2^{1/0}(x_2)$</td>
<td>$&lt; 1e-2$</td>
<td>$&lt; 1e-3$</td>
</tr>
<tr>
<td>$\mu_2^{1/0}(x_3)$</td>
<td>$&lt; 1e-2$</td>
<td>$&lt; 1e-2$</td>
</tr>
<tr>
<td>$\mu_3^{2}(x_1)$</td>
<td>-1.758</td>
<td>0.211</td>
</tr>
<tr>
<td>$\mu_3^{2}(x_2)$</td>
<td>-1.752</td>
<td>0.210</td>
</tr>
<tr>
<td>$\mu_3^{2}(x_3)$</td>
<td>-1.752</td>
<td>0.195</td>
</tr>
</tbody>
</table>

Notes: (i) The terms "$< 1e-2$" and "$< 1e-3$" within the "Bias" column mean that the absolute value of the bias is smaller than $1e-2$ and $1e-3$, respectively. (ii) The simulated panel data have 5,000 cross-sectional observations.

decision process. Though the structural parameters in data generation process are time invariant, we will not use this condition in estimation. The cross-sectional sample size $n$ is 5,000 throughout the numerical studies.

In the first setting, we let $d_x = 3, d_z = 4$ and $m_s = 3$. We randomly generated 50 sets of transition matrices $(F^0, F^1)$, and for each pair of transition matrices, we run the closed form linear estimation in equation (4.1) with 300 replications. So there are 15,000 number of replications in total. In the estimation, we assume both transition matrices and discount factors are known. The estimation results are summarized in table 1. Notice that for the identified period utility difference, the bias is nearly zero. But the partially identified period utility function $\mu^0_2(X_t)$, there is a bias that is constant over the different state values. This is also consistent with what our theory predicts. It is also remarkable that there is substantial decrease in standard deviations when we increase the sampling length from 3 to 4. This corresponds to our claim in section 4 that using $\tilde{v}_{t+1,+} = \delta_t A^T_{t+1,h}b_{t+1}$ rather than $\tilde{v}_{t+1,+} = A^T_{h,t}b_t$ can improve the estimation accuracy.

In the second setting, we let $d_x = 30, d_z = 4$ and $m_s = 3$. Again we randomly generated 50 sets of transition matrices, and run 300 replications for each pair of transition matrices. For this setting, we only simulate panel data with length of 4 periods. Two estimation procedures are used for this setting. The first
Table 2: Estimation of Period Utility Functions: $d_x = 30, d_z = 4$

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Direct</th>
<th>Smooth</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_1^{1/0}(x_{0.25})$</td>
<td>0.011, 0.305</td>
<td>&lt; 1e-2, 0.116</td>
</tr>
<tr>
<td>$\mu_1^{1/0}(x_{0.50})$</td>
<td>&lt; 1e-2, 0.299</td>
<td>&lt; 1e-2, 0.107</td>
</tr>
<tr>
<td>$\mu_1^{1/0}(x_{0.75})$</td>
<td>0.012, 0.305</td>
<td>0.015, 0.114</td>
</tr>
<tr>
<td>$\mu_2^{1/0}(x_{0.25})$</td>
<td>&lt; 1e-2, 0.333</td>
<td>&lt; 1e-2, 0.128</td>
</tr>
<tr>
<td>$\mu_2^{1/0}(x_{0.50})$</td>
<td>&lt; 1e-2, 0.322</td>
<td>&lt; 1e-2, 0.113</td>
</tr>
<tr>
<td>$\mu_2^{1/0}(x_{0.75})$</td>
<td>0.015, 0.333</td>
<td>0.017, 0.129</td>
</tr>
<tr>
<td>$\mu_3^0(x_{0.25})$</td>
<td>-1.745, 0.835</td>
<td>-1.738, 0.288</td>
</tr>
<tr>
<td>$\mu_3^0(x_{0.50})$</td>
<td>-1.729, 0.805</td>
<td>-1.735, 0.247</td>
</tr>
<tr>
<td>$\mu_3^0(x_{0.75})$</td>
<td>-1.732, 0.838</td>
<td>-1.729, 0.270</td>
</tr>
</tbody>
</table>

Notes: (i) The term "< 1e-2" within the "Bias" column means that the absolute value of the bias is smaller than 1e-2 and 1e-3, respectively. (ii) The terms $x_{0.25}$, $x_{0.50}$, and $x_{0.75}$ are the 25%, 50% and 75% quantiles of state variable $X$, respectively. (iii) The simulated panel data have 5,000 cross-sectional observations.

Figure 5.1: Confidence interval (95%) of Period Utility Functions: $d_x = 30, d_z = 4$
is to use the estimators in equation (4.1) directly. The second one refines the initial estimates from the first approach by smoothing approximation. To see their performance, we plotted the 95% confidence interval of the period utility difference $\tilde{\mu}^{1/0}$ and the period utility function $\tilde{\mu}^{2}$ for both methods. Clearly, when the state space is large, it is advantageous to use the smooth conditions to improve the raw estimates of period utility functions. In table 2 we also report the bias and standard error for the period utility functions evaluated at the 25%, 50% and 75% quantiles of the support $\mathcal{X}$.

6 Concluding Remarks

The identification and estimation of DPDC models are considered to be complicated and numerically difficult. This paper shows that the identification of DPDC model is indeed equivalent to the identification of a linear GMM system. So the identification and estimation of DPDC models become easy to address. We show how to identify DPDC models under a variety of restrictions. In particular, we show how to identify the DPDC model without normalizing the period utility function of any alternative. This case is particularly important because we show that normalization of period utility functions can usually bias the counterfactual policy predictions. Due to the equivalence to a linear GMM system, we show how to estimate DPDC models using linear estimation approach without using any terminal conditions or assuming the dynamic programming problem is stationary. The implementation of our estimator does not involve any numerical optimization or iteration.

There are two practically important extensions of this paper. First, one can extend the paper by incorporating the unobservable heterogeneity. Several papers, including Kasahara and Shimotsu (2009) and Hu and Shum (2012), have studied the identification of the CCP, when there are unobservable heterogeneity, such as discrete types in Kasahara and Shimotsu (2009). Since our identification of per period utility functions depends on the state transition matrix and the CCP only, one can then identify the type-specific per period utility functions by using the identified type specific CCP (and state transition distributions).

Second, most paper in the literature of DPDC study the identification under the assumption that the distribution of utility shocks are known. Depending on the sensitivity of the parameter estimates on the specification of the error distribution, allowing the error distribution to be unknown could be practically important. Suppose $\mathcal{G}$ is a set of possible distributions of the utility shocks. Each error distribution $G \in \mathcal{G}$ defines a pair of functionals $\phi$ and $\psi$ being used
in our identification arguments. In addition, each pair of functions $\phi$ and $\psi$ will give rise to a set of formulas of the identified structural parameters according to Proposition 6 or other propositions. Therefore, we can explicitly characterize the identified set of the per period utility functions.

A Proofs

Proof of Proposition 7 State transition laws \{f_1(S_t), f_{t+1}(S_{t+1}|S_t, Y_t) : t = 1, \ldots, T - 1\} are obviously identified, and henceforth assumed known. Suppose \{\mu_t^0, \mu_t^1, \delta_t, v_t, v_T^0, v_T^f : t = 1, \ldots, T\} solves equation (3.7).

Let $\tilde{\mu}_t^0(S_t) \neq \mu_t^1(S_t)$ (for each $t = 1, \ldots, T - 1$) and $\tilde{v}_T^0(S_T) \neq v_T^f(S_T)$ be arbitrary bounded functions of $S_t$ and $S_T$, respectively. Also let $\tilde{\delta}_t \neq \delta_t$ be an arbitrary discount factor. Define

\[
\tilde{v}_T^f(S_T) = \{v_T^f(S_T) - v_T^0(S_T)\} + \tilde{v}_T^0(S_T) \\
\tilde{\mu}_t(S_t) = \tilde{\mu}_t^0(S_t) + \psi\{p_T(S_T)\} \\
\tilde{\mu}_t^1(S_t) = \mu_t^1 + \delta_t \mathbb{E}_{t+1}^{1/0}\{v_{t+1}(S_{t+1})|S_t\} + \mu_t^0(S_t) - \delta_t \mathbb{E}_{t+1}^{1/0}\{\tilde{v}_{t+1}(S_{t+1})|S_t\} \\
\tilde{v}_t(S_t) = \tilde{\mu}_t^0(S_t) + \delta_t \mathbb{E}_{t+1}^{0/1}\{\tilde{v}_{t+1}(S_{t+1})|S_t\} + \psi\{p_t(S_t)\}.
\]

Note that the above construction implies that $\tilde{v}_T^f(S_T) \neq \tilde{v}_T^0(S_T)$, $\tilde{\mu}_t(S_t) \neq \mu_t^1(S_t)$ for each $t$. The above defined parameters \{\tilde{\mu}_t^0, \tilde{\mu}_t^1, \tilde{\delta}_t, \tilde{\phi}_t, \tilde{\psi}_T, \tilde{\phi}_T\} : t = 1, \ldots, T\} also solve equation (3.7). So we conclude that the only identified parameters are \{f_1(S_t), f_{t+1}(S_{t+1}|S_t, Y_t) : t = 1, \ldots, T - 1\}. \qed

Proof of Proposition 6 Starting by $t = T - 1$, we have

\[
\phi\{p_{T-1}(s_{T-1})\} = \mu_T^{1/0}(s_{T-1}) + \delta_{T-1} \mathbb{E}_T^{1/0}\{v_T(S_T)|s_{T-1}\}, \\
\phi\{p_{T-1}(s'_{T-1})\} = \mu_T^{1/0}(s'_{T-1}) + \delta_{T-1} \mathbb{E}_T^{1/0}\{v_T(S_T)|s'_{T-1}\}.
\]

It follows from $\mu_T^{1/0}(s_{T-1}) = \mu_T^{1/0}(s'_{T-1})$ that

\[
\phi\{p_{T-1}(s_{T-1})\} - \phi\{p_{T-1}(s'_{T-1})\} = \delta_{T-1}\mathbb{E}_T^{1/0}\{v_T(S_T)|s_{T-1}\} - \mathbb{E}_T^{1/0}\{v_T(S_T)|s'_{T-1}\},
\]

which then leads to

\[
\delta_{T-1} = \frac{\phi\{p_{T-1}(s_{T-1})\} - \phi\{p_{T-1}(s'_{T-1})\}}{\mathbb{E}_T^{1/0}\{v_T(S_T)|s_{T-1}\} - \mathbb{E}_T^{1/0}\{v_T(S_T)|s'_{T-1}\}},
\]

when $\mathbb{E}_T^{1/0}\{v_T(S_T)|s_{T-1}\} - \mathbb{E}_T^{1/0}\{v_T(S_T)|s'_{T-1}\} \neq 0$. Given $\delta_{T-1}$, we then have $v_{T-1} = \mu_{T-1}^0(S_{T-1}) + \delta_{T-1} \mathbb{E}_T^{0/1}\{v_T(S_T)|S_{T-1}\} + \psi\{p_{T-1}(S_{T-1})\}$. Then the
identification of $\delta_{T-2}$ follows from similar derivations for identifying $\delta_{T-1}$. The other discount factors can be identified similarly.

Proof of Proposition 3. With the stationarity condition, we can drop time subscript "t" from all parameters. We have

$$v = \mu^0 + \delta F^0 v + \psi(p),$$

from equation (3.8), which then implies that

$$(I - \delta F^0)v = \mu^0 + \psi(p).$$

Since $\delta < 1$ and $F^0$ is a transition matrix (hence its largest eigenvalue is one), we conclude that $(I - \delta F^0)$ is invertible. So $v = (I - \delta F^0)^{-1}\{\mu^0 + \psi(p)\}$. Once, $v$ is uniquely determined. The other parameters, $\mu^1/0$, $v^0$ and $v^1$, are determined uniquely by equation (3.8).

Lemma A.1. Let $A$ be an $m \times n$ real matrix with $m \geq n - 1$. Suppose each row of $A$ sums to be zero and rank $A = n - 1$. Suppose linear equation $Ax = b$ has solutions. Then the solution set is $\{A^+ b + c \times 1_n : \forall c \in \mathbb{R}\}$, where $A^+$ is the Moore-Penrose pseudoinverse of $A$, and $1_n$ is a $n$-dimensional vector of ones.

Proof. We know that the solution set of equation $Ax = b$ is $\{A^+ b + (I - A^+ A)a : \forall a \in \mathbb{R}^n\}$. It suffices to show that $(I - A^+ A)$ is an $n \times n$ matrix, whose elements are identical.

Let $A = U \Sigma V^T$ be an singular value decomposition (SVD) of matrix $A$. We know that $A^+ = V \Sigma^+ U^T$, where $\Sigma^+$ is the pseudoinverse of $\Sigma$. Because $U$ and $V$ are both orthogonal matrices, we have $A^+ A = V \Sigma^+ \Sigma V^T$ as an eigenvalue decomposition (EVD). When rank $A = n - 1$, we have

$$\Sigma^+ \Sigma = \begin{bmatrix} I_{n-1} & 0 \\ 0 & 0 \end{bmatrix},$$

where $I_{n-1}$ is $(n - 1) \times (n - 1)$ identity matrix. So the columns of $V$ are eigenvectors of $A^+ A$ corresponding to the eigenvalues 1 and 0. Because the sum of columns of $A$ is zero, $1_n$ is an eigenvector of $A^+ A$ corresponding to eigenvalue zero, and $n^{-1/2} \times 1_n$ is one column of $V$. Removing the column $n^{-1/2} \times 1_n$ from matrix $V$, we obtain an $n \times (n - 1)$ matrix $\tilde{V}$ and $A^+ A = V \Sigma^+ \Sigma V^T = \tilde{V} \tilde{V}^T$. 38
As $V$ is an orthogonal matrix, we have

$$I = VV^T = \begin{bmatrix} \tilde{V} & n^{-1/2} \times 1_n \end{bmatrix} \begin{bmatrix} \tilde{V}^T \\ n^{-1/2} \times 1_n^T \end{bmatrix} = \tilde{V} \tilde{V}^T + n^{-1} 1_n 1_n^T = A^+ A + n^{-1} 1_{n \times n}.$$

Here $1_{n \times n}$ is a $n \times n$ matrix whose elements are all 1. So we have $I - A^+ A = n^{-1} 1_{n \times n}$, and the lemma will follow.

**Lemma A.2.** Let $A_1$ and $A_2$ both be $m \times n$ real matrices with $m \geq 2(n - 1)$. Define a block matrix $A \equiv \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$. For each $i = 1, 2$, suppose each row of $A_i$ sums to be zero, and rank $A = 2n - 2$. Suppose linear equation

$$A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = b$$

has solutions. Let

$$\begin{bmatrix} x_{1,+} \\ x_{2,+} \end{bmatrix} = A^+ b.$$

Then the solution set of the equation is

$$\left\{ \begin{bmatrix} x_{1,+} + c_1 1_n \\ x_{2,+} + c_2 1_n \end{bmatrix} : c_1, c_2 \in \mathbb{R} \right\}.$$

**Proof.** The proof is similar to the proof of lemma [A.1](#). The solution set of equation $Ax = b$ is $\{ A^+ b + (I - A^+ A) a : \forall a \in \mathbb{R}^n \}$. Let $A = U \Sigma V^T$ be an SVD of matrix $A$. We have $A^+ A = V \Sigma^+ \Sigma V^T$ as an EVD. Because rank $A = 2n - 2$ and the row sums of each $A_i$ ($i = 1, 2$) are zero, we have

$$\Sigma^+ \Sigma = \begin{bmatrix} I_{2n-2} \\ 0_{2 \times 2} \end{bmatrix}.$$

So $V$ has two columns $w_1^T = n^{-1/2} (1_n^T, 0_n^T)$ and $w_2^T = n^{-1/2} (0_n^T, 1_n^T)$, because they are two orthonormal eigenvectors corresponding to eigenvalue 0. Removing $w_1$ and $w_2$ from the columns of matrix $V$, we obtain an $2n \times (2n - 2)$ matrix $\tilde{V}$ whose columns are eigenvectors corresponding to the $2n - 2$ nonzero eigenvalues. We then have $A^+ A = V \Sigma^+ \Sigma V^T = \tilde{V} \tilde{V}^T.$

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As $V$ is an orthogonal matrix, we have

$$I = VV^T = \begin{bmatrix} \tilde{V} & w_1 & w_2 \end{bmatrix} \begin{bmatrix} \tilde{V}^T \\ w_1^T \\ w_2^T \end{bmatrix} = \tilde{V}\tilde{V}^T + w_1 w_1^T + w_2 w_2^T = A^+ A + \begin{bmatrix} 1 & \cdots & 1 \\ \cdots & \cdots & \cdots \\ 1 & \cdots & 1 \end{bmatrix}_{n \times n}.$$

The rest of the proof follows immediately.

Proof of Proposition 7. Consider the following two linear system of equations

$$A_t \begin{bmatrix} \bar{v}_t \\ \bar{v}_{t+1} \end{bmatrix} = b_t \text{ and } A_{t-1} \begin{bmatrix} \bar{v}_{t-1} \\ \bar{v}_t \end{bmatrix} = b_{t-1},$$

from which we can identify the period utility function difference $\mu_1^{1/0}$. By Proposition 6, we have two representations of $\mu_1^{1/0}$ from the above two systems,

$$\mu_1^{1/0} = \phi \left( p_{t-1} \right) - \delta_{t-1} F_t^{1/0} A_{t,h}^+ b_t,$$
$$\mu_1^{1/0} = \phi \left( p_{t-1} \right) - F_t^{1/0} A_{t-1,l}^+ b_{t-1}.$$

So we have $\delta_{t-1} F_t^{1/0} A_{t,h}^+ b_t = F_t^{1/0} A_{t-1,l}^+ b_{t-1},$ or

$$F_t^{1/0} \left\{ \delta_{t-1} A_{t,h}^+ b_t - A_{t-1,l}^+ b_{t-1} \right\} = 0.$$

If rank $F_t^{1/0} = d_s - 1$, we have

$$\delta_{t-1} A_{t,h}^+ b_t - A_{t-1,l}^+ b_{t-1} = c \times 1_{d_s},$$

for some $c \in \mathbb{R}$.

Define a $(d_s - 1) \times d_s$ matrix $L$ such that

$$L = \begin{bmatrix} 1 & -1 & \cdots & -1 \\ 1 & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 1 & \cdots & \cdots & 1 \end{bmatrix}_{(d_s - 1) \times d_s}.$$
For a $d_s$-dimensional real vector $x \in \mathbb{R}^{d_s}$, we have $Lx = 0_{d_s}$ if and only if $x = c \times 1_{d_s}$ for some $c \in \mathbb{R}$. Multiplying both sides of equation (A.1) with matrix $L$, we have

$$\delta_{t-1} L A_{t,h}^+ b_t - L A_{t-1,l}^+ b_{t-1} = 0_{d_s}. \tag{A.2}$$

It follows from the definition of $b_t$ and $b_{t-1}$,

$$b_t = \begin{bmatrix} \delta_{t-1}^{-1} M \phi(p_{t-1}) \\ M \phi(p_t) \\ M \psi(p_t) \end{bmatrix} \text{ and } b_{t-1} = \begin{bmatrix} \delta_{t-2}^{-1} M \phi(p_{t-2}) \\ M \phi(p_{t-1}) \\ M \psi(p_{t-1}) \end{bmatrix},$$

that equation (A.2) becomes

$$\left( L A_{t,h}^+ \right) \begin{bmatrix} M \phi(p_{t-1}) \\ \delta_{t-1}^{-1} M \phi(p_{t-1}) \\ \delta_{t-1} M \psi(p_t) \end{bmatrix} - \left( L A_{t-1,l}^+ \right) \begin{bmatrix} \delta_{t-2}^{-1} M \phi(p_{t-2}) \\ M \phi(p_{t-1}) \\ M \psi(p_{t-1}) \end{bmatrix} = 0_{d_s}. \tag{A.3}$$

To rephrase the above display as an equation about $\delta_{t-1}$ and $\delta_{t-2}^{-1}$, write $A_{t,h}^+$ and $A_{t-1,l}^+$ as block matrices,

$$A_{t,h}^+ \equiv \begin{bmatrix} A_{t,h,1}^+ & A_{t,h,2}^+ & A_{t,h,3}^+ \end{bmatrix},$$

$$A_{t-1,l}^+ \equiv \begin{bmatrix} A_{t-1,l,1}^+ & A_{t-1,l,2}^+ & A_{t-1,l,3}^+ \end{bmatrix}.$$

Then equation (A.3) is equivalently written as follows,

$$B_t \begin{bmatrix} \delta_{t-1} \\ \delta_{t-2}^{-1} \end{bmatrix} = h_t,$$

where $B_t$ is a block matrix with dimension $(d_s - 1) \times 2$,

$$B_t = \begin{bmatrix} L A_{t,h,2}^+ M \phi(p_t) + L A_{t,h,3}^+ M \psi(p_t) & -L A_{t-1,l,1}^+ M \phi(p_{t-2}) \end{bmatrix}, \tag{A.4}$$

and $h_t$ is a $(d_s - 1)$-dimensional vector,

$$h_t = L A_{t-1,l,2}^+ M \phi(p_{t-1}) + L A_{t-1,l,3}^+ M \psi(p_{t-1}) - L A_{t,h,1}^+ M \phi(p_{t-1}).$$

If rank $B_t = 2$, we can solve $\delta_{t-1}$ and $\delta_{t-2}^{-1}$ uniquely.

\[\square\]

**Proof of Proposition 8.** The key observation is that when $Z_t$ is time invariant,
the state transition matrix $F_t^d$ is a $d_x$-by-$d_x$ block matrix,

\[
F_t^d = \begin{bmatrix}
  D_t(x_1, x_1) & D_t(x_1, x_2) & \cdots & D_t(x_1, x_{d_x}) \\
  D_t(x_2, x_1) & D_t(x_2, x_2) & \cdots & D_t(x_2, x_{d_x}) \\
  \vdots & \vdots & \ddots & \vdots \\
  D_t(x_{d_x}, x_1) & D_t(x_{d_x}, x_2) & \cdots & D_t(x_{d_x}, x_{d_x})
\end{bmatrix},
\]

of which each element $D_t(x_i, x_j)$ is a $d_z$-by-$d_z$ diagonal matrix, for each period $t$ and each choice $d$. The diagonal matrix $D_t(x_i, x_j)$ has the following form,

\[
D_t(x_i, x_j) = \begin{bmatrix}
  f_t(x_j, z_1|z_i, Y_{t-1} = d) \\
  \vdots \\
  f_t(x_j, z_{d_z}|z_i, Y_{t-1} = d)
\end{bmatrix},
\]

because $f_t(x_j, z_k|z_i, Y_{t-1} = d) = 0$ whenever $z_k \neq z_i$. Let $e_i$ be an $d_z$-dimensional vector whose elements are all zero excepting for the $i$-th element being 1. One can verify that \( \tilde{e}_1 \equiv 1_{2d_t} \otimes e_1, \ldots, \tilde{e}_{d_z} \equiv 1_{2d_z} \otimes e_d_z, \tilde{e}_{d_z+1} \equiv (0^t_{d_z}, 1^T_{d_z})^T \),

belong to the null space of $A_t$, and are linearly independent. Hence, if rank $A_t = 2d_t - d_z - 1$, we have $\mathcal{N}(A_t) = \text{span}(\tilde{e}_1, \ldots, \tilde{e}_{d_z+1})$. Then the solution set for equation (3.20) is

\[
(A.5) \left\{ \begin{bmatrix}
  \tilde{v}_{t,+} \\
  \tilde{v}_{t+1,+}
\end{bmatrix} + \lambda_1 \tilde{e}_1 + \cdots + \lambda_{d_z+1} \tilde{e}_{d_z+1} : (\lambda_1, \ldots, \lambda_{d_z+1}) \in \mathbb{R}^{d_z+1} \right\}.
\]

Let

\[
\tilde{v}_t = \begin{bmatrix}
  \tilde{e}_{i,h} \\
  \tilde{e}_{i,l}
\end{bmatrix}.
\]

Then $\mu_{t-1}^{1/0}$ and $\mu_t^{1/0}$ are identified because $F_t^{1/0} \tilde{v}_{i,h} = F_{t+1}^{1/0} \tilde{v}_{i,l} = 0_{d_z}$ for each $i = 1, \ldots, d_z + 1$. The period utility function $\mu_t^0$ is identified up to an additive constant because for any $(\tilde{v}_{t+1}^0, \tilde{v}_{t+1}^0)$ belonging to the solution set (A.5), we have

\[
\tilde{v}_{t}^0 - F_{t+1}^0 \tilde{v}_{t+1}^0 = \left( \tilde{v}_{t,+} - F_{t+1}^0 \tilde{v}_{t+1,+} \right) + 1_{d_z} \otimes (\lambda_{d_z+1} \times 1_{d_z})
= \left( \tilde{v}_{t,+} - F_{t+1}^0 \tilde{v}_{t+1,+} \right) + \lambda_{d_z+1} \times 1_{d_z}.
\]

So the conclusion of Proposition 6 can be established for the permanent excluded variable case. The same proof of Proposition 7 can be used to identify the
discount factors in the permanent excluded variable situation.

Proof of Proposition 9. We are going to use

\[
\begin{align*}
\mu_{t+1}^{1/0} + F_{t+1}^{1/0} \bar{v}_{t+1} &= \phi(p_t) \\
\mu_t^{1/0} + \delta_{t-1} F_t^{1/0} \mu_t &= \phi(p_{t-1}) \\
\mu_t^{1/0} + \delta_{t-2} F_{t-1}^{1/0} \mu_{t-1} &= \phi(p_{t-2}) ,
\end{align*}
\]

\[
\begin{align*}
\bar{v}_t - \mu_t^0 &- F_{t+1}^0 \bar{v}_{t+1} = \psi(p_t) \\
\bar{v}_{t-1} - \mu_t^0 &- \delta_{t-1} F_t^0 \bar{v}_t = \psi(p_{t-1})
\end{align*}
\]

We have a system

\[(A.6) \quad C_t \begin{bmatrix} \bar{v}_{t+1} \\ v_t \\ \bar{v}_{t-1} \end{bmatrix} = h_t ,
\]

where \(C_t\) and \(h_t\) are as defined in equation (3.22).

Note that the two vectors

\[
b_1^T = (1_{d_s}, 0_{d_s}, -1_{d_s}), \\
b_2^T = (0_{d_s}, 1_{d_s}, (1 + \delta_{t-1}) \times 1_{d_s}),
\]

are linearly independent, and \(C_t b_1 = C_t b_2 = 0_{3d_s}\). So \(b_1\) and \(b_2\) are contained in the null space of matrix \(C_t\), denoted by \(N(C_t)\). By the assumption that \(\text{rank } C_t = 3d_s - 2\), we then have \(N(C_t) = \text{span}(b_1, b_2)\). So the solution set of equation \((A.6)\) is

\[
\{ C_t^+ h_t + \lambda_1 b_1 + \lambda_2 b_2 : \lambda_1, \lambda_2 \in \mathbb{R} \}.
\]

We then have

\[
\mu_t^{1/0} = 3^{-1} \{ \phi(p_t) + \phi(p_{t-1}) + \phi(p_{t-2}) - \left[ F_{t+1}^{1/0} \bigg| \delta_{t-1} F_t^{1/0} \bigg| \delta_{t-2} F_{t-1}^{1/0} \right] C_t^+ h_t \},
\]

is unique. Here

\[
\left[ F_{t+1}^{1/0} \bigg| \delta_{t-1} F_t^{1/0} \bigg| \delta_{t-2} F_{t-1}^{1/0} \right]
\]

is \(1 \times 3\) block matrix. And the identified set of \(\mu_t^0\) is

\[
\{ \mu_t^0 + c \times 1_{d_s} : c \in \mathbb{R} \},
\]
where
\[ \tilde{\mu}_0 = 2^{-1} \left\{ -F_{t+1} \begin{bmatrix} I_{s_t} - \delta_{t-1} F^0_t \end{bmatrix} I_{s_t} \right\} C_t^+ h_t - \psi(p_t) - \psi(p_{t-1}) \].

Then proposition is proved.  

B Counterfactual Policy Predictions under Normalization of Period Utility Functions

For simplicity, we focus on the case where state transition matrices and discount factors are known and time invariant. Let \( F^0 \) and \( F^1 \) be the state transition matrices that generate the observed data. And let \( \delta \) be discount factor. Consider a counterfactual experiment that changes state transition matrices but do not change period utility functions. Let \( F^0_c \) and \( F^1_c \) the state transition matrices under counterfactual experiment. We are interested in predicting the counterfactual CCP.

B.1 Consequence for stationary DPDC models

Suppose the agent’s dynamic programming problem is stationary. One way to identify the stationary DPDC model is to impose restriction 3 so that the period utility function \( \mu^0 \) and discount factor \( \delta \) are both known. Let \( \mu_0 \) and \( \tilde{\mu}_0 \) be two normalized period utility functions. And let \( \tilde{\mu}_0 = (\mu_0(s_1), \ldots, \mu_0(s_{d_s}))^T \) and \( \tilde{\mu}_0 = (\tilde{\mu}_0(s_1), \ldots, \tilde{\mu}_0(s_{d_s}))^T \). Given \( (\mu_0, \delta, F^0_c, F^1_c) \), we will have one counterfactual CCP \( p_c(S) \). Similarly, the set \( (\tilde{\mu}_0, \delta, F^0_c, F^1_c) \) also defines a counterfactual CCP \( \tilde{p}_c(S) \). Write \( p_c = (p_c(s_1), \ldots, p_c(s_{d_s}))^T \) and \( \tilde{p}_c = (\tilde{p}_c(s_1), \ldots, \tilde{p}_c(s_{d_s}))^T \). The following proposition answers the question when will \( p_c = \tilde{p}_c \), so the normalization of period utility function \( \mu^0 \) will be innocuous for predicting counterfactual policy effects.

Proposition B.1. Define \( A \equiv \delta F^1_c / (I - \delta F^0_c)^{-1} \) and \( A_c \equiv \delta F^1_c / (I - \delta F^0_c)^{-1} \). One necessary condition for \( p_c = \tilde{p}_c \) is that \( (\mu_0 - \tilde{\mu}_0) \in N(A - A_c) \), where \( N(A - A_c) \) is the null space of matrix \( A - A_c \). One sufficient condition for \( p_c = \tilde{p}_c \) is that \( \mu^0 - \tilde{\mu}^0 \) equals to a vector of which all entries are identical, and this condition would also be necessary if \( \text{rank}(A - A_c) = d_s - 1 \).

Proof. Given \( \mu^1 \) and \( \tilde{\mu}_0 \), the difference between the ASVF of two alternatives,
We know that
\[ v^1 - v^0 = \mu^{1/0} + \delta F^{1/0} \]

Similarly, the difference between the counterfactual ASVF of two alternatives, \( v^1_c - v^0_c \), is

\[ v^1_c - v^0_c = \mu^{1/0} + \delta F^{1/0}(I - \delta F^0)^{-1} \left\{ \mu^0 + \psi(p_c) \right\} \]

We know that \( v^1 - v^0 = \phi(p) \) and \( v^1_c - v^0_c = \phi(p_c) \). So we conclude

\[ \phi(p) - \phi(p_c) = (A - A_c) \mu^0 + A \psi(p) - A_c \psi(p_c). \]

We have similar conclusion for using the period utility functions \( \hat{\mu}^0 \):

\[ \phi(p) - \phi(\tilde{p}_c) = (A - A_c) \hat{\mu}^0 + A \hat{\psi}(p) - A_c \hat{\psi}(\tilde{p}_c). \]

Hence, we have

\[ \phi(\tilde{p}_c) - \phi(p_c) = \left\{ \phi(p) - \phi(p_c) \right\} - \left\{ \phi(p) - \phi(\tilde{p}_c) \right\} = (A - A_c) \left( \mu^0 - \hat{\mu}^0 \right) - A_c \hat{\psi}(p_c) + A_c \psi(\tilde{p}_c). \]

In other words,

\[ \left\{ \phi(\tilde{p}_c) - A_c \psi(\tilde{p}_c) \right\} - \left\{ \phi(p_c) - A_c \psi(p_c) \right\} = (A - A_c) \left( \mu^0 - \hat{\mu}^0 \right). \]

Define a mapping \( g : \mathbb{R}^{d_s} \to \mathbb{R}^{d_s} \) such that \( g(x) \equiv \phi(x) - A_c \psi(x) \) for any \( x \in \mathbb{R}^{d_s} \). We then have

\[ g(\tilde{p}_c) - g(p_c) = (A - A_c) \left( \mu^0 - \hat{\mu}^0 \right). \]

By the mean value theorem for vector valued mappings, we have

\[
\left[ \int_0^1 \nabla g \left\{ p_c + \tau (\tilde{p}_c - p_c) \right\} \, d\tau \right] (\tilde{p}_c - p_c) = (A - A_c) \left( \mu^0 - \hat{\mu}^0 \right).
\]
Suppose \( \tilde{p}_c = p_c \). We must have \((A - A_c)\left(\mu^0 - \tilde{\mu}^0\right) = 0\), that is \(\mu^0 - \tilde{\mu}^0\) belongs to the null space of \(A - A_c\). When \(\text{rank}(A - A_c) = d_s - 1\), the null space of \(A - A_c\) contains only the vectors, whose elements are identical. This proves the necessary part of the proposition.

For any \(\tilde{\mu}^0 = \mu^0 + a\), where \(a\) is a vector whose elements are all \(a\), we have \(\tilde{v} = v + (1 - \delta)^{-1}a\) and \(\tilde{\mu} = v_c + (1 - \delta)^{-1}a\). Because \(F^{1/0}v = F^{1/0}\tilde{v}\), we have \(\tilde{\mu}^{1/0} = \mu^{1/0}\). Then we have \(\tilde{v}^1 - \tilde{v}^0 = \tilde{v}_c^1 - \tilde{v}_c^0\) for \(F^{1/0}\tilde{v}_c = F^{1/0}\tilde{v}_c\), which implies that \(\tilde{p}_c = p_c\). This shows the sufficiency part.

**B.2 Consequence for DPDC models with finite horizon**

Suppose the agent’s dynamic programming problem has finite horizon, and the last sampling period \(T\) is the decision horizon \(T_s\). We have shown the identification under the assumption that the period utility function associated with one is known (restriction \([1]\)). Let \(\mu^0_t\) and \(\tilde{\mu}^0_t\) be two assumed period utility of alternative \(0\). Let \(p_{t,c}\) and \(\tilde{p}_{t,c}\) be the counterfactual CCP vectors associated with the assumed period utility functions \(\mu^0_t\) and \(\tilde{\mu}^0_t\). The following proposition answers the question when will \(\tilde{p}_{T-1,c} = \tilde{p}_{T-1,c}\) (\(p_{T,c} = \tilde{p}_{T,c}\) is always true). Of course, the proposition can be extended to cover the other periods at the expense of more complicated notations.

**Proposition B.2.** Define \(B \equiv \delta F^{1/0}\) and \(B_c \equiv \delta F^{1/0}_c\). One necessary condition for \(\tilde{p}_{T-1,c} = \tilde{p}_{T-1,c}\) is that \(\left(\mu^0_T - \tilde{\mu}^0_T\right) \in \mathcal{N}(B - B_c)\), where \(\mathcal{N}(B - B_c)\) is the null space of matrix \(B - B_c\). One sufficient condition for \(\tilde{p}_{T-1,c} = \tilde{p}_{T-1,c}\) is that \(\left(\mu^0_T - \tilde{\mu}^0_T\right)\) equals to a vector of which all entries are identical, and this condition would also be necessary if \(\text{rank}(B - B_c) = d_s - 1\).

**Proof.** We first have \(\phi(p_t) = \mu^0_T = \tilde{\mu}^{1/0}_T\), \(v_T = \mu^0_T + \psi(p_T)\) and \(\tilde{v}_T = \tilde{\mu}^0_T + \psi(\tilde{p}_T)\) for the last period \(T\). Next, it follows from \(v^1_{T-1} - v^0_{T-1} = v^1_{T-1} - v^0_{T-1}\) that

\[
\mu^{1/0}_T + \delta F^{1/0}v_T = \tilde{\mu}^{1/0}_T + \delta F^{1/0}\tilde{v}_T,
\]

which can implies that

\[
\mu^{1/0}_T - \tilde{\mu}^{1/0}_T = \delta F^{1/0} \left(\tilde{\mu}^0_T - \mu^0_T\right).
\]

Now consider the counterfactual experiment. We have \(p_{T,c} = \tilde{p}_{T,c} = p_T\) because the counterfactual experiment does not change period utilities. So
\( v_{T,c} = \tilde{v}_T \) and \( \tilde{v}_{T,c} = \tilde{v}_T \). For period \( T - 1 \), however, we have

\[
\phi \left( p_{T-1,c} \right) = \tilde{v}_{T-1,c}^1 - \tilde{v}_{T-1,c}^0 = \mu_{T-1}^{1/0} + \delta F_{c}^{1/0} \tilde{v}_T
\]

and

\[
\phi \left( \tilde{p}_{T-1,c} \right) = \tilde{v}_{T-1,c}^1 - \tilde{v}_{T-1,c}^0 = \tilde{\mu}_{T-1}^{1/0} + \delta F_{c}^{1/0} \tilde{v}_T.
\]

Then

\[
\phi \left( p_{T-1,c} \right) - \phi \left( \tilde{p}_{T-1,c} \right) = \left( \mu_{T-1}^{1/0} - \tilde{\mu}_{T-1}^{1/0} \right) + \delta F_{c}^{1/0} \left( \tilde{v}_T - \tilde{v}_T \right)
\]

\[
= \delta F_{c}^{1/0} \left( \tilde{v}_T - \tilde{v}_T \right) - \delta F_{c}^{1/0} \left( \tilde{\mu}_{T}^{0} - \mu_{T}^{0} \right)
\]

\[
= \delta \left( \tilde{F}_{c}^{1/0} - F_{c}^{1/0} \right) \left( \tilde{\mu}_{T}^{0} - \mu_{T}^{0} \right).
\]

The above display is similar to equation (B.1). So we can apply the arguments in the proof of Proposition B.1 to prove the present proposition. \( \square \)

**References**


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