Playing Anonymous Games
Using Simple Strategies

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Anonymous Games

• $n$ players, $k = O(1)$ strategies

• Payoff of each player depends on
  • Her identity and strategy
  • The number of other players who play each of the strategy
  • **NOT** the identity of other players
Anonymous Games
Nash Equilibrium

• Players have no incentive to deviate

• $\epsilon$-Approximate Nash Equilibrium ($\epsilon$-ANE):
  Players can gain at most $\epsilon$ by deviation
Previous Work

$\epsilon$-ANE of $n$-player $k$-strategy anonymous games:

- [DP’08]: First PTAS $n^{(k/\epsilon)^O(k^3)}$
- [CDO’14]: PPAD-Complete when $\epsilon = 2^{-n^c}$ and $k = 5$
How small can $\epsilon$ be so that an $\epsilon$-ANE can be computed in polynomial time?
<table>
<thead>
<tr>
<th></th>
<th>Running time</th>
<th>$\epsilon$</th>
<th># of strategies</th>
</tr>
</thead>
<tbody>
<tr>
<td>[DP’08a]</td>
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<td>$k &gt; 2$</td>
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<td>[GT’15]</td>
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<td>[DKS’16a]</td>
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Our Results

Fix any \( k > 2, \delta > 0 \)

- First poly-time algorithm when \( \epsilon = \frac{1}{n^{1-\delta}} \)
- A poly-time algorithm for \( \epsilon = \frac{1}{n^{1+\delta}} \) \( \implies \) FPTAS
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Fix any $k > 2, \delta > 0$

• First poly-time algorithm when $\epsilon = \frac{1}{n^{1-\delta}}$

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• A faster algorithm that computes an $\epsilon \approx \frac{1}{n^{1/3}}$ equilibrium
Anonymous Games

• Player $i$’s payoff when she plays strategy $a$

\[ u^i_a(\Pi_{n-1}) = 0.3 \quad u^i_a(\Pi_{n-1}) = 0.7 \]

• $u^i_a$: $\Pi_{n-1}^k \rightarrow [0, 1]$

• $\Pi_{n-1}^k = \{(x_1, \ldots, x_k) \mid \Sigma_i x_i = n - 1\}$
Poisson Multinomial Distributions

• $k$-Categorical Random Variable ($k$-CRV) $X_i$ is a vector random variable $\in \{k\text{-dimensional basis vectors}\}$

• An $(n, k)$-Poisson Multinomial Distribution (PMD) is the sum of $n$ independent $k$-CRVs $X = \sum X_i$
Player 1 plays strategy 1

Player 2 plays strategy 1 or 2

Player 3 plays strategy 2 or 3
Poisson Multinomial Distributions
Poisson Multinomial Distributions (PMDs)

= Sum of independent random (basis) vectors

= Mixed strategy profiles of anonymous games
Better understanding of PMDs

Faster algorithms for $\epsilon$-ANE of anonymous games
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Pure Nash Equilibrium

Player 1

Player 2

\vdots

Strategy 1

Strategy 2

Strategy 3

Player n
Lipschitz Games

• An anonymous game is $\lambda$-Lipschitz if

$$\left| u^i_a (\cdot) - u^i_a (\cdot) \right| \leq \lambda \| \cdot \|_1$$

• [DP’15, AS’13] Every $\lambda$-Lipschitz $k$-strategy anonymous game admits a $(2k\lambda)$-approximate pure equilibrium
Lipschitz Games

• \((2k\lambda)\)-approximate pure equilibrium

Bad case: \(\lambda = 1\)

\[ u_a^i(\lambda) = 0 \quad u_a^i(\lambda) = 1 \]
\( u_a^i( ) = 0 \)

\( u_a^i( ) = 1 \)
$$d_{TV}(P, Q) := \frac{1}{2} \cdot \|P - Q\|_1$$

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Smoothed Game \([\text{GT’15}]\)

- Given a game \(G\), construct a new game \(G_\delta\)

\[
u'(\cdot) = \mathbb{E}[u(\cdot)]
\]

- \(G_\delta\) is \(\tilde{O}\left(\frac{1}{\sqrt{n\delta}}\right)\)-Lipschitz
\( \tilde{O}(1/n^{1/3}) \)-ANE in Polynomial Time

- A \((2k\lambda)\)-ANE of \(G_\delta\) is a \((2k\lambda + \delta)\)-equilibrium of \(G\)
  - Gain at most \(2k\lambda\) by switching to \((1 - \delta, \frac{\delta}{k-1}, \ldots, \frac{\delta}{k-1})\)
  - Gain at most \((2k\lambda + \delta)\) by switching to \((1, 0 \ldots 0)\)

- \(\lambda = \tilde{O}\left(\frac{1}{\sqrt{n\delta}}\right) \Rightarrow \epsilon = \tilde{O}\left(\frac{1}{\sqrt{n\delta}}\right) + \delta = \tilde{O}\left(\frac{1}{n^{1/3}}\right)\)
\[ d_{TV}(\mathcal{N}(\mu_1, \Sigma_1), \mathcal{N}(\mu_2, \Sigma_2)) \leq \tilde{O}\left(\frac{1}{\sqrt{n\delta}}\right) \| - \|_1 \]

- Size-free multivariate Central Limit Theorem [DKS’16]: an \((n, k)\)-PMD is \(\text{poly}(k/\sigma)\) close to discrete Gaussians

- Two Gaussians with similar mean and variance are close
Our Results

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$O(1/n^{0.99})$-ANE
Quasi-PTAS when $\epsilon = 0(1/n^c)$

• Small $d_{TV} \Rightarrow$ Similar payoffs

• Limitation:
  • Cover-size lower bound [DKS’16]: even when $k = 2$
    Any proper $\epsilon$-cover $S$ must have $|S| \geq n \left(1/\epsilon\right)^{\Omega(\log(1/\epsilon))}$
\( \epsilon = 1/n^{1/3} \)

Two moments

\( \log(1/\epsilon) \) moments

\( \epsilon = 1/n^{0.99} \)

\( \Theta(1) \) moments
Moment Matching Lemma

• For two PMDs to be $\epsilon$-close in $d_{TV}$
  
  [DP’08, DKS’16] need first $\log(1/\epsilon)$ moments to match

• We provide quantitative tradeoff between
  
  • The number of moments we need to match
  • The size of the variance

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Moment Matching Lemma

• Multidimensional Fourier transform
  • Exploit the sparsity of the Fourier transform

• Taylor approximations of the log Fourier transform
  • Large variance $\Rightarrow$ Truncate with fewer terms
$O(1/n^{0.99})$-ANE in Polynomial Time

• There always exists an equilibrium with variance
  \[\epsilon n = n^{-0.99} \cdot n = n^{0.01}\]

• Construct a poly-size $\epsilon$-cover of large variance PMDs
  • Polynomial-size: Match only degree $O(1)$ moments
\[ |S| \geq n \left(\frac{1}{\epsilon}\right)^{\Omega(\log(1/\epsilon))} \]
\[ X_i = \left( (1 - p)e_j + pe_{-j} \right) \]

\[ X = \sum X_i \]

\[ |S| \leq n^{k-1} \]

\[ 1/n^{1/3} \text{-ANE} \]

\[ \frac{1}{n^{0.99}} \text{-ANE} \]

\[ \text{PMDs with large variance} \]

\[ |S| = n^k O(1/(1-0.99)) \]
Conclusion

Computing $\epsilon$-ANE of $n$-player anonymous games

• First poly-time algorithm when $\epsilon = \frac{1}{n^{1-\delta}}$
  • New moment-matching lemma for PMDs

• A poly-time algorithm for $\epsilon = \frac{1}{n^{1+\delta}} \implies$ FPTAS
Open Problems

FPTAS?

ε = 1

ε = 1/n

ε = 1/n^c

ε = 1/2^{n^c}