Playing Anonymous Games Using Simple Strategies

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Anonymous Games

• $n$ players
• $k$ actions

• Payoff of each player depends on
  • Her identity and strategy
  • The number of other players playing each strategy
  • **NOT** the identity of other players
Anonymous Games
Anonymous Games

• $n$ players, $k = O(1)$ actions
• Player $i$’s payoff when he plays strategy $a$

\[ u^i_1(\cdot) = 0.7 \quad u^i_2(\cdot) = 0.3 \quad u^i_3(\cdot) = 1 \]

• $u^i_a$: $\Pi^k_{n-1} \to [0, 1]$
• $\Pi^k_{n-1} = \{(x_1, \ldots, x_k) | \sum_i x_i = n - 1\}$
Approximate Nash Equilibria (ANE)

• A mixed strategy is a distribution on \([k] = \{1 \ldots k\}\)

• A mixed strategy profile \((s_1 \ldots s_n)\) is an \(\varepsilon\)-ANE iff

\[
\forall i \in [n], \forall a' \in [k], \quad \mathbb{E}_{x \sim s_{-i}} \left[u^i_{a'}(x)\right] \leq \mathbb{E}_{x \sim s_{-i}, a \sim s_i} \left[u^i_a(x)\right] + \varepsilon
\]
Previous Work

Computing $\epsilon$-ANE of $n$-player $k$-strategy anonymous games

- [DP’07]: First PTAS $\text{poly}(n) \cdot \left(\frac{1}{\epsilon}\right)^{\frac{1}{\epsilon}k!}$
- [CDO’14]: PPAD-Complete when $\epsilon = 2^{-n^c}$ and $k = 5$
How small can $\epsilon$ be so that an $\epsilon$-ANE can be computed in polynomial time?
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<td>$n^{\text{poly}(k)} \cdot (1/\epsilon)^k \log(1/\epsilon)^{O(k)}$</td>
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Our Results

Fix any $k > 2, \delta > 0$

- First poly-time algorithm when $\epsilon = \frac{1}{n^{1-\delta}}$
- A poly-time algorithm for $\epsilon = \frac{1}{n^{1+\delta}} \implies \text{FPTAS}$
Our Results

Fix any $k > 2, \delta > 0$

- First poly-time algorithm when $\epsilon = \frac{1}{n^{1-\delta}}$
- A poly-time algorithm for $\epsilon = \frac{1}{n^{1+\delta}} \Rightarrow$ FPTAS
- A faster algorithm that computes an $\epsilon \approx \frac{1}{n^{1/3}}$ equilibrium
Poisson Multinomial Distributions

• $k$-Categorical Random Variable ($k$-CRV) $X_i$
  • A mixed strategy $s_i \iff a$-CRV: $\Pr[X_i = e_a] = s_i(a)$

• An $(n, k)$-Poisson Multinomial Distribution (PMD) is the sum of $n$ independent $k$-CRVs $X = \sum X_i$
  • A mixed strategy profile $\iff$ an $(n, k)$-PMD
Finding a Pure Equilibrium

• Guess histogram

• Build a bipartite graph
  • Players \([n]\) on one side, strategies \([k]\) on the other side
  • \((i, a) \in E\) iff player \(i\) can play action \(a\) given the histogram
  • Is there a max flow of value \(n\)?
Lipschitz Games

• An anonymous game is $\lambda$-Lipschitz if

$$\forall i \in [n], \forall a \in [k], \forall x, y \in \Pi_{n-1}^k, \quad |u^i_a(x) - u^i_a(y)| \leq \lambda \|x - y\|_1$$

• [DP’15, AS’13] Every $\lambda$-Lipschitz $k$-strategy anonymous game admits a $(2k\lambda)$-approximate pure equilibrium
Lipschitz Games

- \((2k\lambda)\)-approximate pure equilibrium

\[\forall i \in [n], \forall a \in [k], \forall x, y \in \prod_{n-1}^{k}, \quad |u_a^i(x) - u_a^i(y)| \leq \lambda \|x - y\|_1\]

\[u_a^i(\text{green}) = 0 \quad \quad u_a^i(\text{blue}) = 1\]
\( u^i_a( ) = 0 \)

\( u^i_a( ) = 1 \)
\[ u^i_a(\cdot) = 0 \]

\[ u^i_a(\cdot) = 1 \]

\[ d_{TV}(P, Q) := \frac{1}{2} \cdot \|P - Q\|_1 \]

\[ d_{TV} = 1 \]

\[ d_{TV} \ll 1 \]
Smoothed Game \[\text{[GT'15]}\]

- Given a game \( G \), construct a new game \( G_\delta \)

\[ u'(\cdot) = \mathbb{E}[u(\cdot)] \]

- \( G_\delta \) is \( \tilde{O}\left(\frac{1}{\sqrt{n\delta}}\right) \)-Lipschitz
$\tilde{O}(1/n^{1/3})$-ANE in Polynomial Time

• A $(2k\lambda)$-ANE of $G_\delta$ is a $(2k\lambda + \delta)$-equilibrium of $G$
  
  • Gain at most $2k\lambda$ by switching to $\left(1 - \delta, \frac{\delta}{k-1}, \ldots, \frac{\delta}{k-1}\right)$
  
  • Gain at most $(2k\lambda + \delta)$ by switching to $(1, 0 \ldots 0)$

• $\lambda = \tilde{O}\left(\frac{1}{\sqrt{n\delta}}\right) \Rightarrow \epsilon = \tilde{O}\left(\frac{1}{\sqrt{n\delta}}\right) + \delta = \tilde{O}\left(\frac{1}{n^{1/3}}\right)$
\(\tilde{O}(1/n^{1/3})\)-ANE in Polynomial Time

\[ d_{TV}(\cdot, \cdot) \leq \frac{1}{\sqrt{n\delta}} \| \cdot - \|_1 \]

- Size-free multivariate Central Limit Theorem [DKS’16]: an \((n, k)\)-PMD is \(\text{poly}(k/\sigma)\) close to discrete Gaussians
- Two Gaussians with similar mean and variance are close
$O(1/n^{0.99})$-ANE
Quasi-PTAS when $\epsilon = O(1/n^c)$

• Let $(X, d)$ be a metric space

  Given $\epsilon > 0$, a subset $Y \subseteq X$ is a proper $\epsilon$-cover of $X$ if for every $X \in X$ there exists some $Y \in Y$ such that
  \[ d(X, Y) \leq \epsilon \]

• Small $d_{TV} \Rightarrow$ similar payoffs
Quasi-PTAS when $\epsilon = 0(1/n^c)$

- For computing pure equilibrium, we guess histograms

$\mathcal{S} = \{ \ldots \}$
Quasi-PTAS when $\epsilon = O(1/n^c)$

- Guess this PMD from an $\epsilon$-cover
  - Compute the $\epsilon$-best responses of each player
  - Reconstruct the PMD from these best responses

- Limitation
  - Cover-size lower bound [DKS’16]: even when $k = 2$
    Any proper $\epsilon$-cover $S$ must have $|S| \geq n (1/\epsilon)^{\Omega(\log(1/\epsilon))}$
$O(1/n^{0.99})$-ANE
Moment Matching Lemma

• [DP’08, DKS’16] need first $\log(1/\epsilon)$ moments to match

• For discrete Gaussians, the first two moments are enough

• We provide quantitative tradeoff between
  • The number of moments we need to match
  • The size of the variance
Moment Matching Lemma

• Fix $0 < c < 1$ and let $\epsilon = n^{-c}$. Let $X, Y$ be two $(n, k)$-PMDs. Let $\Sigma$ and $\Sigma'$ be the covariance matrix of $X$ and $Y$. If all the non-zero eigenvalues of $\Sigma$ and $\Sigma'$ are at least $(\epsilon n/k)$, and $\forall \ell \leq \frac{2+2c}{1-c}$, all the parameter moments of degree $\ell$ satisfy $|M_m(X) - M_m(Y)| \leq \epsilon$. Then we have $d_{TV}(X, Y) \leq \epsilon$.

For $m = (m_1, \ldots, m_k) \in \mathbb{Z}_+^k$, $M_m(X) \overset{\text{def}}{=} \sum_{i=1}^n \prod_{j=1}^k p_{i,j}^{m_j}$.
Moment Matching Lemma

- Multidimensional Fourier transform
  - Exploit the sparsity of the Fourier transform

- Taylor approximations of the log Fourier transform
  - Large variance $\Rightarrow$ Truncate with fewer terms
$0(1/n^{0.99})$-ANE in Polynomial Time

• There always exists an equilibrium with variance
  \[ \epsilon n = n^{-0.99} \cdot n = n^{0.01} \]

• Construct a poly-size $\epsilon$-cover of large variance PMDs
  • [DP’08, DKS’16] need the first $\log(1/\epsilon)$ moments to match
  • We only match the first $O(1)$ moments
All $(n, k)$-PMDs

$|\mathcal{S}| \geq n \left(\frac{1}{\varepsilon}\right)^{\Omega(\log(1/\varepsilon))}$
\[ X_i = ((1 - p) e_j + p e_{-j}) \]
\[ X = \sum X_i \]

\[ \frac{1}{n^{1/3}} \text{-ANE} \]

PMDs with large variance

\[ \frac{1}{n^{0.99}} \text{-ANE} \]
Our Results

• First poly-time algorithm when $\epsilon = \frac{1}{n^{1-\delta}}$

• A poly-time algorithm for $\epsilon = \frac{1}{n^{1+\delta}} \Rightarrow$ FPTAS

Open Problems

FPTAS for Nash in anonymous games?
Thanks!

Q & A