Query Complexity of Approximate Nash Equilibria

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Joint work with

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Nash Equilibrium

<table>
<thead>
<tr>
<th>Audience</th>
<th>Presenter</th>
<th>Put effort into presentation (E)</th>
<th>Do not put effort (NE)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pay attention (A)</td>
<td>2, 2</td>
<td>−8, −10</td>
<td></td>
</tr>
<tr>
<td>Do not pay attention (NA)</td>
<td>0, −3</td>
<td>0, 0</td>
<td></td>
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</table>

• Players cannot improve their payoffs by deviation:

(A, E)  (NA, NE)  ((1/5 A, 4/5 NA), (4/5 E, 1/5 NE))
Multi-Player Games

• $n$-player two-strategy games
  • Normal form: description size $= n2^n$

• Query complexity
  • Oracle access to payoff functions
ANE vs. WSNE

• $\epsilon$-approximate Nash equilibrium (ANE)
  • Players can gain at most $\epsilon$ by unilateral deviation

• $\epsilon$-well-supported Nash equilibrium (WSNE)
  • Any strategy that is used with non-zero probability by a player must be an $\epsilon$-best response
ANE vs. WSNE
Recap

• Input: An $n$-player two-strategy game $G$ (payoff oracle)

• Output: An $\epsilon$-ANE ($\epsilon$ is a small constant in this talk)

• Goal: Minimize the number of queries
  • $2^n$ queries gives full information
## Related Work

<table>
<thead>
<tr>
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<th>Solution Concept</th>
<th># of Queries</th>
</tr>
</thead>
<tbody>
<tr>
<td>[Hart and Nisan ’13]</td>
<td>$(1/2)$-ANE (deterministic) exact NE</td>
<td>$2^\Omega(n)$</td>
</tr>
<tr>
<td>[Babichenko ’14]</td>
<td>$\varepsilon$-WSNE</td>
<td>$2^\Omega(n)$</td>
</tr>
<tr>
<td>Our Result</td>
<td>$\varepsilon$-ANE</td>
<td>$2^\Omega(n/\log n)$</td>
</tr>
</tbody>
</table>
[Babichenko ’14]: $QC_p(WSNE(n, \epsilon)) = 2^{\Omega(n)}$

- $p$ = success probability $= 2^{-\Theta(n)}$
Challenge: WSNE vs. ANE

If everyone plays sub-optimally with probability $\epsilon$, then in aggregate about $\epsilon n$ players are playing arbitrarily, making the outcome of the game unpredictable.
Our Reduction

$$WSNE(n, \varepsilon) \leq ANE(n, \varepsilon) \leq ANE(8n \log(n/\varepsilon), \varepsilon/8)$$
Our Reduction

$$\text{WSNE}(n, \epsilon) \leq \text{ANE}(8n \log(n/\epsilon), \epsilon/8)$$

$$\text{QC}_p(\text{WSNE}(n, \epsilon)) = 2^{\Omega(n)} , \quad p = 2^{-\Theta(n)}$$

$$\text{QC}_p(\text{ANE}(n, \epsilon)) = 2^{\Omega(n/ \log n)} , \quad p = 2^{-\Theta(n/ \log n)}$$
Our Reduction

\[ WSNE(n, \varepsilon) \leq ANE(8n \log(n/\varepsilon), \varepsilon/8) \]

- Given a game \( G \), construct \( G' \):
  - Replace each player with \( O(\log n) \) agents
  - Use majority voting to decide the strategy of each group
  - Payoffs are determined using payoff functions of \( G \)
Our Reduction

- In an \((\epsilon/4)\)-ANE of \(G'\)
  - If strategy 0 is not \(\epsilon\)-best response for player \(i\) \(\implies\)
    - Each agent in group \(i\) can put probability at most 1/4 on 0
  - \(O(\log(n/\epsilon))\) Players + Tail bounds
    \(\implies\) \(\Pr[\text{Majority of group } i = \text{strategy 0}] \leq \epsilon/n\)

- Set all \(\leq \epsilon/n\) probabilities to zero
Open Problems

- $2^{\Omega(n/ \log n)}$?
  - $2^{\Omega(n)}$ [Rubinstein ’16]

- Is there a reduction with $O(n)$ players?
  - Fault-tolerant computation: Simulate an $n$-gadget circuit with “cheap” gadgets that malfunction with probability $\epsilon$