# Maximizing Network Utilization with Max-Min Fairness in Wireless Sensor Networks

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*Abstract*—**The state of the art for optimal data-gathering in wireless sensor networks is to use additive increase algorithms to achieve fair rate allocation while implicity trying to maximize network utilization. We explicitly formulate the problem of maximizing the network utilization subject to a max-min fair rate allocation constraint in the form of two coupled linear programs. We first show how the max-min rate can be computed efficiently for a given network. We then adopt a dual-based approach to maximize the network utilization. The analysis of the dual shows the sub-optimality of previously proposed additive increase algorithms with respect to bandwidth efficiency. Although in theory a dual-based sub-gradient search algorithm can take a long time to converge, we find empirically that setting shadow prices to 1 results in near-optimal solutions within one iteration (within 2% of the optimum in 99.65% of the cases). This results in a fast heuristic distributed algorithm that has a nice intuitive explanation — rates are allocated sequentially after rank ordering flows based on the number of downstream receivers whose bandwidth they consume.**

## I. INTRODUCTION

We consider the problem of efficient and fair rate allocation for data gathering applications in wireless sensor networks. The radio communication bandwidth resources available on many wireless sensor network platforms are fundamentally constrained. Hence, efficient bandwidth utilization is imperative for maximizing the amount of data extracted and minimizing the delay in extracting this data. At the same time, fairness is also a key consideration in wireless sensing applications in order to maintain a balanced view of the sensor environment (for instance, fair data gathering is essential for reducing the estimation error in field reconstruction).

Additive increase-based mechanisms for rate control are popular in the context of wired networks. This is because they are optimal for lexicographic fairness [1], as well as for other notions of fairness such as proportional fairness [2]. The popularity of additive increase algorithms in wireline networks have also led to their adaptation to rate allocation in wireless sensor networks [9], [11].

The main difference between wireless networks and wireline networks is that flows in a wireless network not only consume bandwidth *usefully* on the links they are active on but also *wastefully* on links that they interfere with. Moreover, there is heterogeneity in the amount of interference (i.e., bandwidth wastage) that each flow may cause. This fundamental difference between wired and wireless networks demands a fresh look at the problem of fair and efficient rate control algorithms for wireless networks in general and wireless sensor networks specifically.

In wireless settings, a fair rate allocation may treat equally flows that cause high interference as well as flows that cause less interference. On the contrary, a rate allocation that favors flows causing less interference may be able to provide higher network utilization (as measured by the total sum of the flow rates). Hence, there can be a fundamental tension between fairness and efficiency in wireless networks [3]. Consequently, the additive increase approaches that provide lexicographic fairness even in the context of wireless networks, are not well suited from the perspective of bandwidth efficiency. In this work, to address both fairness and efficiency goals, instead of looking at lexicographic fairness, we define the objective as maximizing the network utilization while ensuring that the rate allocations satisfy a slightly weaker notion of max-min fairness.

We model the problem as follows: There are $n$ sources in the network that are trying to send data to a single sink over a given tree. Every source has a shortest path through one or more intermediate nodes to the sink. Every receiver in the network has limited bandwidth. The objective of the problem is to maximize the sum of the source rates subject to a constraint of max-min fair rate allocation. We define a rate allocation to be max-min fair if the minimum rate allocated to any flow is the maximum over all possible rate allocations.

We formulate the above problem as two coupled linear programs — the first problem identifies the max-min rate allocation, while the second maximizes the sum-rate subject to the constraint determined by the solution of the first problem. We prove that the optimal solution to the first problem is the minimum of ratios of available bandwidths to upstream demands. This characterization allows for the efficient solution of the first problem via a tree-based aggregation and dissemination. We analyze the second problem using Lagrange duality. The analysis of the dual reveals the sub-optimality of additive increase mechanisms. Although solving the dual problem using sub-gradient search techniques can potentially result in slow convergence, we find empirically that initializing all shadow prices to one provides near-optimal results within one iteration. This gives a fast near-optimal distributed heuristic (which provides solutions within 2% of the optimum in

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99.65% of the cases) that has an intuitive explanation — flows from sources are scheduled sequentially after rank ordering them on the number of downstream receivers whose bandwidth they consume (either directly or via interference).

This paper is organized as follows: In section II we present our receiver bandwidth capacity model that will be essential in modeling the interference constraints in our optimization problem. Using the bandwidth capacity model we formulate the problem of maximizing network utilization while allocating a max-min fair rate as two coupled linear programs. In section III we present a lemma that helps us calculate the max-min rate in a tree. In section IV we present an example to motivate our claim that additive increase algorithms while providing a max-min fair rate allocation but do not maximize network utilization. In section V we take a dual based approach to design a near optimal heuristic for our problem. We also use the dual to present a more rigorous proof for the sub-optimality of the additive increase algorithms. In section VI we present simulation results to highlight the performance of our algorithm. In section VII we present the related work pertinent to this problem. Finally in section VIII we present our conclusions and the future for this work.

## II. MODELING RECEIVER BANDWIDTH CONSUMPTION IN WIRELESS NETWORKS

In this section we present a model that captures the bandwidth consumption at a receiver in a tree $\mathbf{T}$ rooted at the sink. The essence of the model is that it captures the interference observed by a receiver. This model is identical to the one proposed by us in [9] and is similar to the one used by Rangwala *et al.* [11] to capture the effects of interference. We denote the set of all communication links in the network by the set $\mathbf{E}$, and the set of all nodes by the set $\mathbf{V}$. Every receiver in the network has a finite receiver bandwidth capacity given by the set $\mathbf{B}$. The routing tree rooted at the sink is denoted by $\mathbf{T} \subset \mathbf{G}$ where $\mathbf{G}$ is the communication graph.

Due to the broadcast nature of wireless links, any flow from a child $i$ to its parent $j$ on the tree $T$ consumes bandwidth on all receivers that are neighbors of $i$ on the graph $G$ (we assume here that the neighbor set captures all interfering nodes, and therefore refer to the edges in $E$ that are not part of $T$ as noise edges). It is this feature that makes the problem of rate allocation in these networks very different from that observed on a wired network.

We illustrate our model with an example. Figure 1 shows a 6 node topology. The solid lines indicate a parent child relationship in the tree. The dashed line represents the existence of a link between two neighbors. Since neighbors do not send any useful data to each other this link is referred to as a noise link. For each source, any rate consumed by the source on the link with its parent would result in consumption of an equal rate on the noise links. Thus the rate at which node 2 sends its data to node 1 would be equal to the bandwidth consumed at the receiver node 3. Based on our model the constraint on the rates at node 3 would be as follows:

$$r_{noise}^{(2)} + r_{noise}^{(3)} + r_{src}^{(6)} \leq B^{(3)} \tag{1}$$

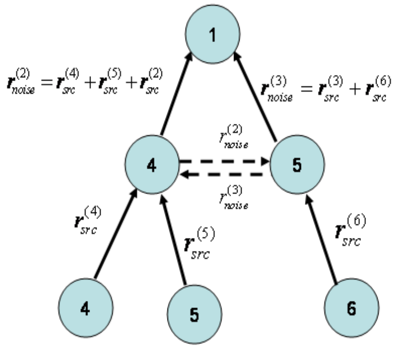


Fig. 1. A 6 node topology: An illustrative of example of the receiver bandwidth model

where $B^{(3)}$ is the receiver capacity of node 3 and $r_{src}^{(6)}$ is the source rate of node 6. $r_{noise}^{(2)}$ and $r_{noise}^{(3)}$ are the output rates at node 2 and node 3 respectively and are given by:

$$r_{noise}^{(2)} = r_{src}^{(2)} + r_{src}^{(4)} + r_{src}^{(5)}$$

and

$$r_{noise}^{(3)} = r_{src}^{(3)} + r_{src}^{(6)}$$

The radios are assumed to be half duplex *i.e.*, a node cannot send and receive simultaneously. To account for the half duplex nature of the radio, the term $r_{src}^{(6)}$ appears twice in equation 1. Once independently to account for the consumption of bandwidth during reception at node 3 and once as part of the term $r_{noise}^{(3)}$ to account for the forwarding of the flow originating at node 6.

In general the receiver capacity constraint at a node $i$ can be given as follows:

$$\sum_{j \in C^{(i)}} r_{src}^{(j)} + \sum_{j \in N^{(i)}} \sum_{k \in C^{(j)}} r_{src}^{(k)} + \sum_{j \in N^{(i)}} r_{src}^{(j)} \leq B^{(i)} \tag{2}$$

Where $N^{(i)}$ is the set of all neighbors of $i$. The half duplex assumption implies that $i \in N^{(i)}$. $C^{(i)}$ is the set of all nodes $j$ that have $i$ in its path to the sink. $r_{src}^{(j)}$ represents the rate at which data generated at node $j$ is being transmitted.

Using the receiver bandwidth model we can now formulate the maximization of the network capacity utilization while maintaining max-min fairness as two coupled constrained optimization problems $\mathbf{P1}$ and $\mathbf{P2}$. The variables used in our formulation are described as follows:

- $\mathbf{R_{src}}$: An $N \times 1$ vector representing the rate allocated to each source $i \in \mathbf{V}$
- $\mathbf{N}$: An $N \times N$ matrix representing the presence of a noise edge $n_{ij} \in N$ between two nodes $i, j \in \mathbf{V}$
- $\mathbf{C}$ :An $N \times N$ matrix that gives the parent-child relationships on the data gathering tree. $c_{ij} \in C^{(i)}$ is 0 if node $i$ is not in node $j^{'s}$ path to the sink and $c_{ij} = 1$ otherwise.
- $\mathbf{R_{in}}$ : An $N \times 1$ vector, representing the total input rate arriving at each node.
- $\mathbf{R_{noise}}$: An $N \times 1$ vector, representing to total output rate exiting from a node.
- $Y$:A scalar, representing the minimum rate among all flows.

The optimization problem is formulated as follows:

$$\begin{array}{ll}\mathbf{P1}: & \\ \max & Y \text{ s.t.} \\ & \mathbf{R_{in}} + \mathbf{N} \times \mathbf{R_{noise}} \preceq \mathbf{B} \\ & \mathbf{R_{in}} = \mathbf{C} \times \mathbf{R_{src}} \\ & \mathbf{R_{noise}} = \mathbf{C} \times \mathbf{R_{src}} + \mathbf{R_{src}} \\ & r_{src}^{(i)} \geq Y \ \forall \ i \ \in \ \mathbf{T} \end{array}$$

$$\begin{array}{ll}\mathbf{P2}: & \\ \max & \sum_{i \in \mathbf{T}} r_{src}^{(i)} \text{ s.t.} \\ & \mathbf{R_{in}} + \mathbf{N} \times \mathbf{R_{noise}} \preceq \mathbf{B} \\ & \mathbf{R_{in}} = \mathbf{C} \times \mathbf{R_{src}} \\ & \mathbf{R_{noise}} = \mathbf{C} \times \mathbf{R_{src}} + \mathbf{R_{src}} \\ & r_{src}^{(i)} \geq Y^* \ \forall \ i \in \mathbf{T} \end{array}$$

The constraints of our optimization problem come directly from our bandwidth consumption model that we had presented before. The problem $\mathbf{P1}$ is the max-min rate problem. The optimal solution $Y^*$ to $\mathbf{P1}$ gives the highest possible minimum rate achievable amongst all possible rate allocation vectors. The problem $\mathbf{P2}$ uses $Y^*$ as a constraint in order to guarantee the best possible minimum rate to all its sources and presents a rate allocation vector that will maximize the sum rate, thus maximizing utilization. In the following section we present a lemma showing that the solution to $\mathbf{P1}$ can be found by taking the minimum of the ratios of the available bandwidths to upstream demands. The algorithm itself can be implemented by using a tree-based aggregation and dissemination mechanism.

## III. CALCULATING THE MAX-MIN SOURCE RATE ON A TREE

The max-min rate is the optimal solution to the problem $\mathbf{P1}$ denoted by $Y^*$. In order to calculate the max-min rate for a given tree we define the term available bandwidth at a receiver ($B_{available}^{(i)}$) as follows:

$$B_{available}^{(i)} = \frac{B^{(i)}}{\Gamma^{(i)}} \tag{3}$$

Where $\Gamma^{(i)}$ is defined as:

$$\Gamma^{(i)} = \sum_{j \in C^{(i)}, i \neq j} c_{ij} + \sum_{j \in N^{(i)}} \sum_{k \in C^{(j)}, k \neq j} c_{jk} + \sum_{j \in N^{(i)}} n_{ij}$$

$\Gamma^{(i)}$ is the sum of the total number of immediate children of node $i$, the total number of neighbors of node $i$ and the total number of children of each of node $i$'s neighbors.

The optimal solution of $\mathbf{P1}$ could be found by observing the available bandwidth $B_{available}^{(i)}$ at each receiver in the network and selecting the minimum of these. The following lemma justifies our claim.

*Lemma 3.1:* The optimal solution $Y^*$ of the primal $\mathbf{P1}$ is the $\min(B_{available}^{(i)}) \ \forall \ i \in \mathbf{V}$.

*Proof:* We define a node $k$ as a bottle neck node if:

$$k = \text{argmin}(B_{available}^{(i)}) \ \forall \ i \in \mathbf{V}$$

- **Case 1:** Assume:

$$Y^* < \min(B_{available}^{(i)}) \ \forall \ i \in \mathbf{V}$$

We can do a rate allocation for all sources $j$ that are children of the bottle neck node $k$ or the children of the neighbor of the bottle neck node $k$ such that $r_{src}^{(j)} = B_{available}^{(k)}$ without violating the bandwidth constraint on node $k$. Since $k$ is the bottle neck node, $r_{src}^{j}$ will be the minimum of all rates allocated to all sources. This implies that we have a rate allocation where

$$\min(r_{src}^{i}), \ \forall \ i \ \in \ V > Y^*$$

Thus we have a contradiction.

- **Case 2:** Assume $Y^* > \min(B_{available}^{(i)}) \ \forall \ i \in V$. Then for node $k$,

$$\sum_{j \in C^{(k)}} r_{src}^{(j)} + \sum_{g \in N^{(k)}} \sum_{z \in C^{(g)}} r_{src}^{(z)} + \sum_{j \in N^{(k)}} r_{src}^{(j)} > B^{(k)}$$

Thus the bandwidth capacity constraint is violated for node $k$.

Hence $Y^* = \min(B_{available}(i)) \ \forall \ i \ \in \ \mathbf{V}$. ■

Based on this lemma a simple algorithm can be developed to calculate the max-min rate on a tree. In order to find the max-min rate every child calculates its available bandwidth ($B_{available}^{(i)}$) and forwards it to the parent. The parent computes the minimum of these and compares it with its own available bandwidth. It then forwards the minimum of these two quantities to its parent. The parent thus performs an aggregation on the available bandwidth in its sub-tree and forwards the minimum to its own parent. Since a parent does not need to calculate its available bandwidth and forward the aggregated minimum available bandwidth to its parent, till it receives information from all its children, the aggregation process proceeds sequentially. Since the aggregation process proceeds sequentially from the leaves of the tree to the root, the algorithm to calculate the max-min rate terminates at the root. The minimum available bandwidth calculated by the root would then be the minimum of all available bandwidths and hence would quantify the max-min rate of the tree. The root can now disseminate this information to every node in the tree by simply sending it downstream over the tree.

## IV. ADDITIVE INCREASE ALGORITHMS AND MAX-MIN FAIRNESS

Currently proposed solutions that achieve max-min fairness while implicitly trying to maximize network utilization( [9], [11]) use the following additive increase mechanism. Sources in the network are allowed to increase their rates equally by a small value $\epsilon$. When a receiver in the network is constrained, it constrains all its neighbors, its neighbors children and its own children. This process continues till the point, when all nodes in the network are constrained. Since all nodes have equal increments and the first node to exhaust its bandwidth would be the bottle neck node, algorithms using additive increase technique would achieve the optimal solution to $\mathbf{P1}$. Even though additive increase algorithms can achieve a solution to $\mathbf{P1}$, while consuming the network capacity, we claim that it will not necessarily achieve a solution for $\mathbf{P2}$. In this section we present insights into our claim through an example and present a more rigorous proof in Section V.

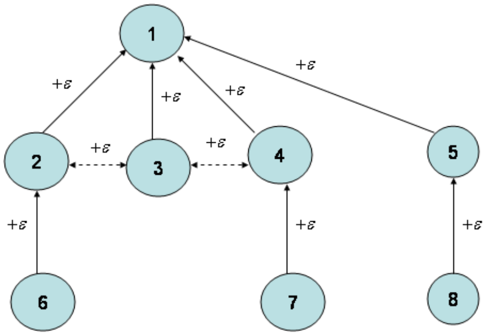


Fig. 2. An example depicting the sub-optimality of the additive increase technique for maximizing network utilization while maintaining a max-min fair rate allocation.

Assume all nodes except node 1 are sources in figure 2. Let node 5 be the bottleneck node. For this topology any increment in the rate of node 3 will consume bandwidth at node 2 and node 4. For e.g. if we increment the rate at node 3 by $\epsilon$ we will be consuming a bandwidth $\epsilon$ at receiver 2, a bandwidth $\epsilon$ at receiver 4 and a bandwidth $\epsilon$ at receiver 1. Thus an increment $\epsilon$ in source rate of node 3 will result in wastage of network capacity equal to $2\epsilon$. A higher throughput could be achieved by simply allocating all nodes the max-min rate and then giving the remaining capacity to nodes 2 and 4. It is easy to see that this allocation would ensure that for increment $\epsilon$ in source rates of 2 and 4 they would not be wasting any bandwidth. The example shows that there exist topologies where additive increase mechanisms might be sub-optimal.

Apart from the sub-optimality another draw back of additive increase algorithms is the estimation of the increment $\epsilon$. In real systems an accurate estimate of $\epsilon$ is critical to avoid oscillations [11]. Moreover the convergence of these algorithms is O($\frac{B}{\epsilon}$) where $B$ is the maximum receiver bandwidth, which implies a trade off between the speed of convergence and the accuracy of the solution depending on the choice of $\epsilon$.

## V. A DUAL BASED APPROACH

In order to gain insights into the dynamics of the problem we plan to adopt a dual based approach. The shadow price interpretation of the Lagrange multipliers [10] from the dual will present us with mechanisms to design distributed algorithms that maximize the network utilization while guaranteeing a max-min fair rate to all sources.

### A. The Lagrange Dual

We introduce Lagrange multipliers in order to relax constraints in the primal $\mathbf{P2}$ to obtain the Lagrange dual function. We will concern ourselves only with the dual of $\mathbf{P2}$ and assume that the optimal max-min rate will be calculated from the primal $\mathbf{P1}$ using lemma 3.1. The Lagrange dual function of the primal $\mathbf{P2}$ is:

$$D(\boldsymbol{\lambda}) = \max_{\mathbf{R_{src}} \succeq \mathbf{Y}^*} \left( \sum_{i \in T} r_{src}^{(i)} - \boldsymbol{\lambda}^{\mathbf{T}} \times ((\mathbf{N} \times (\mathbf{C} + \mathbf{1}) + \mathbf{C}) \times \mathbf{R_{src}} - \mathbf{B}) \right)$$

On expanding the matrix notation we get:

$$\begin{aligned} D(\boldsymbol{\lambda}) \quad &= \max\nolimits_{\mathbf{R_{src}} \succeq \mathbf{Y}^*} (\textstyle\sum_{i \in T} r_{src}^{(i)} \\ &- \textstyle\sum_{i \in T} \lambda_i (\sum_{j \in C^{(i)}} r_{src}^{(j)} \\ &+ \textstyle\sum_{j \in N^{(i)}} \sum_{k \in C^{(j)}} r_{src}^{(k)} \\ &+ \textstyle\sum_{j \in N^{(i)}} r_{src}^{(j)} - B^{(i)})) \end{aligned} \tag{4}$$

We can rearrange equation 4 to obtain:

$$\begin{aligned} D(\lambda) \quad &= \max\nolimits_{\mathbf{R_{src}} \succeq \mathbf{Y}^*} (\textstyle\sum_{i \in T} r_{src}^{(i)} (1 \\ &- (\textstyle\sum_{i \in C^{(j)}} \lambda_j \\ &+ \textstyle\sum_{i \ \in C^{(j)}} \sum_{k \in N^{(j)}} \lambda_k \\ &+ \textstyle\sum_{i \in N^{(j)}} \lambda_j)) + \sum_{i \in T} \lambda_i B^{(i)}) \end{aligned} \tag{5}$$

Since the original problem is a linear program the dual will also be an LP given by

$$\mathbf{D} : \min_{\boldsymbol{\lambda} \succeq \ 0,} D(\boldsymbol{\lambda})$$

Also since the solutions are feasible for both problems the duality gap would be zero [10]. Hence our objective would be to minimize the dual instead of maximizing the primal.

Let

$$\zeta^i(\mathbf{R_{src}^*}) = \sum_{j \in C^i} r_{src}^{(j)*} + \sum_{j \in N^{(i)}} \sum_{k \in C^{(j)}} r_{src}^{(k)*} + \sum_{j \in N^{(i)}} r_{src}^{(j)*}$$

From the Lagrange dual function it can be seen that the sub-gradient w.r.t $\lambda_i$ is:

$$\frac{\partial D}{\partial \lambda_i} = -(\zeta^i(R_{src}^*) - B^{(i)}) \tag{6}$$

Since the dual is a linear program, the objective of minimizing the Lagrange dual can be achieved by tracing the graph in the direction of the negative gradient. We will use the above fact to develop our distributed algorithm.

### B. Analyzing the Dual to Design a Distributed Algorithm

The Lagrange dual function can be rewritten as:

$$\mathbf{D} : \min_{\lambda \succeq 0} \left( \max_{\mathbf{R_{src}} \succeq \mathbf{Y}^*} \left( \sum_{\forall \ i} r_{src}^{(i)} \mu_i \right) + \sum_{\forall \ i} \lambda_i B^i \right)$$

Where $\mu_i$ is given by:

$$\mu_i = 1 - ( \sum_{i \in C^{(j)}} \lambda_j + \sum_{i \in C^{(j)}} \sum_{k \in N^{(j)}} \lambda_k + \sum_{i \in N^{(j)}} \lambda_j) \tag{7}$$

To solve the dual $\mathbf{D}$ we could use sub gradient techniques. Sub gradient techniques are iterative, where at each step $t$ we increment the shadow prices $\lambda_i$ in the direction of the negative gradient as follows:

$$\lambda_i(t+1) = [\lambda_i(t) + \alpha_t(\zeta^i(R_{src}^*) - B^i)]^+ \tag{8}$$

Where $\mathbf{R}^*_{\mathbf{src}}$ are the optimal source rates that solves:

$$\max_{\mathbf{R_{src}} \succeq \mathbf{Y}^*} (\sum_{\forall\ i} r^{(i)}_{src} \mu_i) \tag{9}$$

At every step $t$ we are required to find the $\mathbf{R}^*_{\mathbf{src}}$ that solves equation 9. In order to find a feasible solution for equation 9, assuming $\lambda$ is fixed, we will require to allocate all $r^{(i)}_{src}$ at least the max-min rate $Y^*$. Given that all sources are allocated at least the max-min rate we would require to allocate the maximum available bandwidth to the source $i$ having the highest $\mu_i$. We would then proceed, allocating the remaining bandwidth to the source with the second highest $\mu_i$. We continue allocating bandwidths to sources till all sources have been constrained. Thus bandwidth allocation is based on an ordering of the sources based on their coefficients $\mu_i$. Also, instead of looking at $\mu_i$ for each source $i$ we could assign each source $i$ a weight $w_i$ given by:

$$w_i = \frac{1}{\sum_{i \in C^{(j)}} \lambda_j + \sum_{i \in C^{(j)}} \sum_{k \in N^{(j)}} \lambda_k + \sum_{i \in N^{(j)}} \lambda_j} \tag{10}$$

The ordering, and the prioritization of rate allocation, in order to maximize equation 9, can now be done based on the weights $w_i$ for each source $i$.

To achieve the optimal $\mathbf{D}$ we should be running the sub-gradient algorithm for multiple iterations ($t > 1$), solving the maximization problem in equation 9, until the shadow prices converge. Fortunately through simulations we can show that by setting $\lambda_i = 1, \forall\ i$ in our specific problem, $99.65\%$ of the time we achieve close to $2\%$ of the the optimal in the very first iteration. The details of the simulation and its performance with respect to the optimal are presented in section VI. Thus instead of running the sub-gradient algorithm for multiple iterations, we set the shadow prices $\lambda_i = 1, \forall\ i$ and perform only the first iteration of the sub-gradient algorithm. Our algorithm for maximizing network utilization with max-min fair rate allocation therefore simply consists of optimizing equation 9 by setting the shadow prices to 1. The specifics of the algorithm have been provided in section V-D.

Setting the shadow prices $\lambda_i = 1, \forall\ i$, presents an intuitive explanation to the algorithm. When we set all shadow prices to 1, the weight $w_i$ is inversely proportional to the number of receivers node $i$ interferes with during the transmission of its data to the sink. Thus the ordering suggests that we allocate the maximum bandwidth to nodes that cause the least amount of interference.

### *C. Sub-Optimality of the Additive Increase Algorithms*

In section IV we presented a motivating example for the sub-optimality of additive increase algorithms. Our analysis of the dual in the previous section provides a more quantitative argument for this claim. Primarily it suggests that the rate allocations in the network need to follow an ordering based on the amount of interference that each source generates while transmitting data to the sink. On the contrary, in additive increase algorithms no such ordering exists since all sources are allowed to increment by the same amount. The lack of prioritization in rate allocation is the primary cause for the sub-optimality of additive increase algorithms.

### *D. The Algorithm*

We now present an algorithm for the maximization of equation 9. The algorithm '*Maximization of Network Utilization*' presented in figure 3 proceeds as follows; In the **initialization** phase all sources in the network set their 'CONSTRAINED' flag to 'FALSE'. Every node $i$ calculates its weight $w_i$ using equation 10 and setting the shadow price $\lambda_i = 1,\ \forall\ i$. In order to calculate the weight $w_i$, the node $i$ requires information about the number of parents it has (the number of nodes between itself and the sink), and the number of neighbors of each of its parents and the total number of its neighbors. Each of the three quantities can be obtained by the node during the process of tree formation itself. In effect, every node during the tree formation process, needs to forward the total number of neighbors it possesses and the number of hops to the sink. These two quantities can be used by the nodes to calculate the quantities mentioned above for calculating the weight $w_i$ at the end of the tree formation.

In **step 1**, each node calculates its per node available bandwidth. The bottle neck bandwidth is then the minimum of all the available bandwidths. From lemma 3.1, this bottle neck bandwidth is the max-min rate and hence is allocated to every source in the network. An algorithm to calculate the minimum available bandwidth on a tree using a tree-based aggregation and dissemination mechanism is presented in section III.

In **step 2**, we calculate the pending bandwidth at each node in the network. To calculate its pending bandwidth every receiver notes the total output rate from each of its children and the total output rate from each of its neighbors. The pending bandwidth is then the difference between the bandwidth capacity of the receiver and the sum of the output rates from all its children and its neighbors. For any receiver if the pending bandwidth is negative or zero it constraints all its neighbors their children and its own children. A constrained node can no longer increment its source rate.

In **step 3**, for every node in the network we look at the pending bandwidth at every node that is on the path from the source to the sink, and nodes that are neighbors to these intermediate nodes, and set the pending available bandwidth to the minimum of these. In case the pending available bandwidth is positive, we compare its weight with every other source that is not constrained and increment its bandwidth only if it has the maximum weight. From an implementation perspective, for this step we require that every node has information about the maximum weight currently active in the network. This can be done by pushing the information about the weights to the root and the root then disseminating the maximum weight to all its children. The calculation of the minimum pending bandwidth, described above, can also be done using a tree-based implementation. Every node starting from the root needs to gather its own pending bandwidth and its neighbors pending bandwidth and pass on to its children the minimum of these quantities.

Once a node has incremented its bandwidth (since it was the node with the highest weight), it would become constrained since it would have consumed the maximum available bandwidth in its path. Therefore it would require to remove itself from the list of active sources allowing some other node to become the source with the highest weight. It can perform this operation by informing the root and allowing the root to disseminate this information over the tree.

In **step 4**, we check the constrained flag for all nodes in the network and if all nodes have been constrained the algorithm terminates, else we repeat the algorithm from **step 2**.

While describing the various steps of the algorithm we have presented an implementation perspective to these steps as well. The implementation description gives an operational picture of the algorithm in a real system. This description suggests that although the algorithm is not completely distributed (the decision making is not completely local, it relies on information exchange with the root) it would be more scalable than an implementation where all the computation is done centrally — maintaining the complete topology information centrally and running an LP solver to compute the optimum. By allowing information exchange between the root and the various nodes we have made most of the computation distributed (the pending bandwidth and the weights are calculated locally at the nodes) and reduced the complexity of the computation at the root, for e.g., sorting to obtain the minimum available bandwidth and to obtain the current maximum weight. Our asymptotic analysis of the algorithm suggests that the over head of this information exchange is not high, giving us an acceptable polynomial bound on the number of messages exchanged.

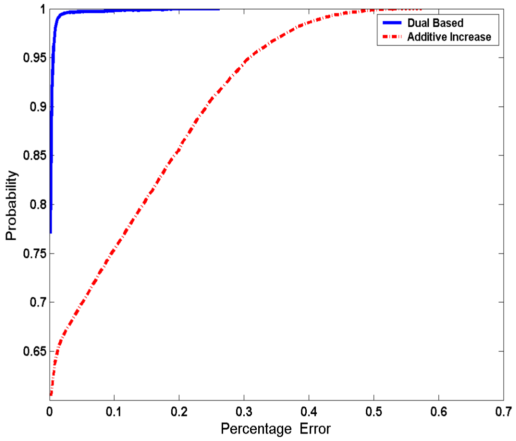
### *E. Asymptotic Bounds for the Dual based Algorithm*

The asymptotic bounds on the dual based algorithm can be calculated as follows: **Step 1** of the algorithm would take $O(n)$ transmissions to calculate the max-min rate. **Step 2** of the algorithm would take $O(n)$ transmissions to calculate the pending bandwidth at each of the intermediate nodes. In **Step 3** of the algorithm once a source node is constrained it needs to populate this information to all nodes in the tree in order to remove itself from the list of source nodes. In order achieve this goal a simple mechanism would be to propagate this information to the root which will collate this information into a new list of sources that are capable of incrementing their bandwidth. This new list could than be propagated to all sources in the tree. Since the total edges in a tree having $n$ nodes are $n-1$, the total number of transmissions to accomplish **Step 3** would be $O(n)$.

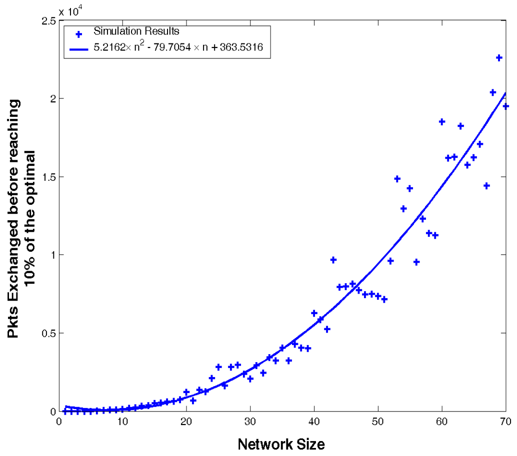
The algorithm terminates when all sources are constrained. Thus **Step 3** and **Step 2** will be executed $O(n)$ times. Hence the algorithm would converge to the solution within $O(n^2)$ transmissions.

## VI. PERFORMANCE EVALUATION

In order to evaluate the performance of our algorithm we choose network sizes ranging from 6 to 70. For each network size we choose 9 instances of trees obtained by running a shortest path algorithm on a random deployment. For each instance of a tree we give every receiver in the tree a bandwidth uniformly chosen between 10 and 250. We choose 20 such bandwidth distributions for each tree. Thus for each network size we have 9 different trees, for each tree there are 20 different instances (each with a different bandwidth distribution) giving a total of 180 instances for each network size. Since our network size ranges from 6 to 70, we have a total of $180 \times 65 = 11700$ different trees for our evaluation.



(a) CDF of the error observed between the optimal throughput achievable and the throughput achieved using dual based algorithm and the additive increase algorithm.



(b) Performance of the algorithm in terms of the number of packets exchanged before achieving 10% of the optimal.

Fig. 4. Performance evaluation of the dual based algorithm and the additive increase algorithm.

In order to evaluate the performance of our algorithm for each of the 11700 instances we generated an LP for the problem **P2**. Using a centralized solver LP SOLVE [19] we obtained the optimal solution for the max-min rate $Y^*$ and the solution to the problem **P2**. We then ran our dual based algorithm and an additive increase algorithm, described in section IV, on each of the 11520 instances to solve the problem **P2** in a distributed manner. Figure 4(a) shows the CDF of the error between the optimal solution from a centralized solver and the solutions obtained from our dual based algorithm, and the additive increase algorithm. For the dual based algorithm the CDF in figure 4(a) shows that for 99.65% of the instances we are able to achieve close to 2% of the optimal throughput. Of the instances that had greater than or equal to 10% error, we were close to 10% of the optimal throughput in 18

```
Algorithm Maximization of Network Utilization:
Initialization
$contrained_i =$ FALSE $\forall\ i$
$\lambda_i = 1\ \forall\ i$
$w_i = \frac{1}{\sum_{i \in C^{(j)}} \lambda_j + \sum_{i \in C^{(j)}} \sum_{k \in N^{(j)}} \lambda_k + \sum_{i \in N^{(j)}} \lambda_j} \forall\ i,\ w_i\ \in\ \mathbf{W}$
[Step 1] max-min Rate:
$B^{(i)}_{available} = \frac{B^{(i)}}{\sum_{j \in C^{(i)}, i \neq j} c_{ij} + \sum_{j \in N^{(i)}} \sum_{k \in C^{(j)}, k \neq j} c_{jk} + \sum_{j \in N^{(i)}} n_{ij}} \forall\ i$
$r^{(i)}_{src} = \min(B^{(i)}_{available})\ \forall\ i$
[Step 2] Pending Bandwidth:
for $\forall\ i$ if $\sum\ c_{ij} \neq 0$ such that $j \in\ C^{(i)}, j \neq i$
    do $B^{(i)}_{pending} = B^i - \zeta^i(B_{src})$
        if $B^{(i)}_{pending} \leq 0$ then
            do Constrain all children, neighbors, and neighbors children.
               Remove constrained nodes from list $\mathbf{W}$.
[Step 3] Updating Source bandwidth:
for $\forall\ i$
    do $pend\_bw = \min(B^{(j)}_{pending}), \forall\ j$ such that $i \in C^j$ or $i \in N^j$ or $k \in N^j,\ i \in C^k$
        if $(w_i == max(\mathbf{W})$ and $constrained_i == FALSE)$ then
            do $r^{(i)}_{src} = r^{(i)}_{src} +\ pend\_bw$
[Step 4] Checking termination condition:
if $(contrained_i$ = TRUE) $\forall\ i$ then end
else goto Step 2
```

Fig. 3. Algorithm for the maximization of network utilization

instances and close to $20\%$ in two of the instances. Figure 4(a) also highlights the sub-optimality of the additive increase algorithms. It shows that in more than $15\%$ of the runs we experienced an error greater than $20\%$, there were $10\%$ of the runs which experienced an error greater than $30\%$ and $5\%$ of the runs experienced an error greater than $40\%$. As highlighted in section IV the sub-optimality of the additive increase algorithm is due to the lack of prioritization of the sources during rate allocation.

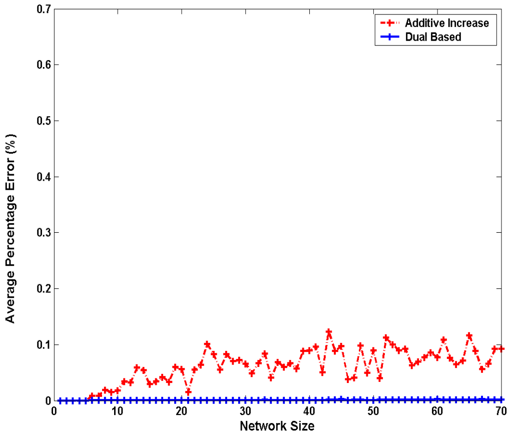
In figure 4(b) we plot the number of packets transmitted before the dual based algorithm converges to a solution. Based on a regression fit we estimate that it grows as O($n^2$). These bounds match the asymptotic bounds that were obtained analytically in section V-E.

Figures 5(a) and 5(b) show the average percentage error and maximum percentage error observed while running the dual based algorithm and the additive increase algorithms across different network sizes. For the dual based algorithm figure 5(a) reiterates the results of figure 4(a) showing that across different network sizes the average percentage error remains very close to zero. The additive increase algorithm however becomes progressively worse as the size of the network is increased. Although the average error exhibited by the additive increase algorithms is not large ($\sim\ 10 - 12\%$), the maximum error exhibited is quite large ($\sim\ 40 - 55\%$). The performance of the additive increase algorithm depends on the placement of the bottleneck node in the topology. If the bottleneck node is very close to the root, in most cases all sources would not be able to get more than the max-min rate even in the optimal solution. In these scenarios the additive increase algorithm would be able to achieve the optimal. However as the bottleneck node starts moving away from the root, the rate distribution among sources would change with a few sources getting very high rates in the optimal solution. Under such a scenario the additive increase algorithm seems to fail. For small networks, since the average diameter of the network is also small, the bottleneck node would be close to the root. However for large networks since the diameter is large, chances of the bottleneck node being farther away from the root are higher leading to an uneven distribution of source rates. This reasoning thus throws light on the performance of the additive increase algorithm as the network size is increased.

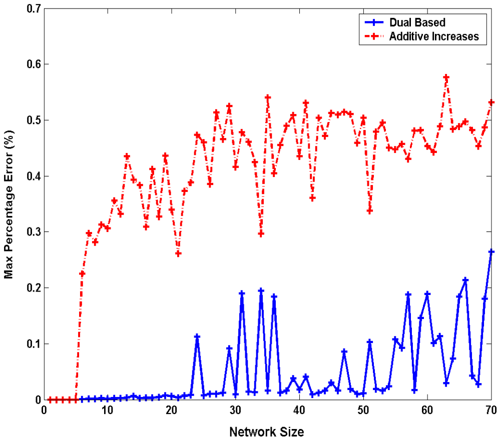
## VII. RELATED WORK

Application of optimization theory to the design and analysis rate control algorithms was first introduced in the wireline context in the seminal paper by Kelly *et al.* [2]. This seminal work established that distributed additive increase-multiplicative decrease rate control protocols can be derived as solutions to an appropriately formulated optimization problem. The application of duality and the sub-gradient approach to solve the same problem was then introduced in the classic work by Low and Lapsley [4]. Since these two works there has been considerable research primarily in the wired context in understanding not only rate control algorithms but network protocols in general and their interaction across multiple layers from the perspective of the optimization problem they aim to solve. A detailed survey of this literature is presented in an article by Chiang *et al.* [5].

For wireless networks, the problem of cross layer optimization has been addressed in the works by Chiang *et al.* [6],



(a) Average percentage error vs network size



(b) Maximum percentage error vs network size

Fig. 5. Performance analysis of the dual based algorithm and the additive increase algorithm on the basis of the percentage error generated for different network sizes.

Johansson *et al.* [7] and Wang *et al.* [8]. These works introduce the dual decomposition technique to address the problem of cross layer optimization in wireless networks and present algorithms for performing joint transport and power control [6] or joint transport and MAC layer design [8].

The problem of rate allocation in particular has been looked at from different perspectives in the domain of wireless networks. Liao *et al.* [16] look at the max-min fair rate allocation problem for packet based wireless access networks. They achieve their goal by assigning the flows at the access point a concave utility function and applying a max-min fair criteria on these flows. However [16] addresses the problem for fixed infrastructure based wireless access networks which is different from our scenario of a data gathering tree in wireless sensor network. Moreover their objective is to maximize the minimum utility of all flows and not to maximize the network utilization.

In [17] Calin *et al.* propose a routing scheme that can optimize resource allocation in a wireless ad-hoc network in order to maximize the aggregate utility of all flows in the network. They use the shadow price interpretation of the dual to present a bidding scheme that allows for a combined routing and rate control heuristic. This work assumes a concave utility function resulting in a notion of proportional fairness. We differ from [17] since in this work we explicitly try to maximize the network utilization while maintaining a weaker notion of the max-min fairness criterion.

Kun *et al.* propose EWCCP [18], a congestion control algorithm for wireless ad-hoc networks designed to provide proportional fairness to flows in the network. The similarity between EWCCP and the algorithm proposed in this work is that congestion signaling in EWCCP explicitly takes into account the interference set of a node while generating a congestion signal for flows traversing that specific node. This is similar to our ordering of the sources for rate allocation based on the amount of interference generated by each source. The difference between the two works is that EWCCP is designed to work within the context of an AIMD protocol, namely TCP, whereas we show in our work that additive increase algorithms are sub-optimal to the joint problem of maximization of network utilization and achieving the highest minimum rate possible. Moreover the notion of fairness achieved by EWCCP is proportional fairness as opposed to max-min fairness.

Wang *et al.* [13], [14] present algorithms for achieving max-min fairness and lexicographic max-min fairness [1] for Aloha random access networks. However the objective of [13] and [14] is to ensure fairness of link rates and not end-to-end flows. In [15] Tassuilas *et al.* present a centralized algorithm for achieving lexicographic max-min fairness in wireless ad hoc networks. Our objective in this work differs from that of [15], since the objective here is to maximize network utilization while maximizing the minimum rate. Moreover our aim is to present a distributed solution as compared to the centralized solution presented in [15].

The problem of max-min fair rate control has been looked at in the context of wireless sensor networks. In an earlier work [9], we presented an additive increase-based rate allocation scheme that guarantees a weaker notion of max-min fairness. In [9] we present a TDMA-based MAC which guarantees a max-min rate allocation by assigning slots to various sources. The number of slots correspond to a source rate that is calculated using an additive increase scheme. Rangwala *et al.* [11] also present an additive increase-multiplicative decrease solution for fair congestion control. The source rates in IFRC are allowed to increase using an additive increase algorithm similar to the one described in section IV. Both these works try to achieve a max-min fair rate allocation while trying to implicitly maximize network utilization. As shown in section IV these techniques are sub-optimal when the dual objective of maximization of network utilization and fairness are taken into consideration.

In the field of wireless sensor networks duality based approaches are not limited to designing and analyzing rate control algorithms. Our approach of analyzing the dual to achieve a distributed solution follows the approach presented by Ye and Ordonez [12], where a distributed dual based gradient search algorithm is proposed for the problem of maximizing data extraction under energy constraints.

## VIII. CONCLUSION AND FUTURE WORK

We have formulated the problem of maximizing network utilization while guaranteeing the best possible minimum rate to sources in a wireless sensor network. We model the problem as two coupled linear programs. By analyzing the dual we are able to show that existing additive increase techniques are provably sub-optimal. Moreover our analysis of the dual results in a heuristic that presents near optimal performance.

There are several directions in which we could extend this work. One of our goals is the implementation of our dual based algorithm on a real sensor network test bed. The objective of such an implementation would be to do a performance comparison with existing rate control mechanisms, such as IFRC [11], that have additive increase algorithms at the core of their design. A real test bed implementation would also help validate the assumptions we have made while modeling the constraints in our problem. In our modeling we have made an implicit assumption that every receiver can hear every interferer that is consuming bandwidth at the receiver. In a real deployment this assumption might be weak since transmitters can cause interference in receivers that are not within range. A real test bed environment would help us ascertain the affects of such phenomenon on the results obtained from our algorithm.

Another direction to pursue is to model the fair and efficient rate control design problem under a different MAC setting. The current modeling may hold for a CSMA or a TDMA based MAC. It would be interesting to see what modifications would be required in modeling the same problem with the assumption of a CDMA based MAC.

In the current problem formulation we have focused on transport layer optimization alone. Interesting extensions to this work include joint transport/routing/MAC/power-control cross-layer design.

Our current work has highlighted an inherent tension between fairness and throughput in wireless networks. An important goal is to characterize the tradeoff between the goals of fairness and throughput in a wireless network. Another important dimension to this problem is the consideration of different notions of fairness.

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